Sub- to supersonic streaming of ions towards the cathode in a beam-generated plasma model

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1. Introduction

The streaming of ions towards the cathode is studied for a steady state beam-generated plasma model. The flow model may have relevance to certain plasma devices. The ionizing beam electrons stream from the cathode towards the anode along a strong magnetic field on a background of neutral particles and bulk- electrons and ions. The plasma in the cathode-anode ionizing range consists of at least two regions that influence the ion-fluid motion: An outer region where the ions are accelerated under influence of a quasineutral electric field, pressure gradients and collisions, and a sheath region close to the cathode where the self-consistent electric field is dominating. A smooth transition between these regions assumes that a Bohm-criterium [1] is fulfilled: The ions have to be supersonic when entering the sheath region. In the model presented a pressure gradient due to temperature drop of ions towards the cathode is necessary for the sub- to supersonic transition of ion flow. Under such an assumption smooth solutions of the governing equations can be followed from the outer region and through the sheath region all the way to the cathode. The sub- to supersonic transition solution is found using a technique due to Bilicki et. al., [2].

2. Basic model and equations

2.1 Particle conservation equations, outer region

The model has been set up to describe in particular the ion flow in the 'ionizing' region where all charged particles predominately flow along magnetic field lines of constant length L, connecting the cathode and the anode. A one-dimensional model is therefore possible. We introduce characteristic scales and use non-dimensional variables for convenience:

$$x = \frac{x}{L},\ \rho_\alpha = \frac{n_\alpha}{n_{pl}},\ u_\alpha = \frac{v_\alpha}{v_{cs}},\ \Phi = \frac{e}{\kappa T_e}\phi,$$

$$\varepsilon = -\frac{eL}{\kappa T_e}E,\ f_{\alpha\beta} = \frac{L}{v_{cs}}v_{\alpha\beta},\ \tau_i = \frac{T_i}{T_e}.$$

Here $v_{cs} = (\frac{\kappa T_e}{m_i})^{\frac{1}{2}}$ is the ion-acoustic speed, $v_{\alpha\beta}$ are collision frequencies and n_{pl} a characteristic plasma density. The beam-, the bulk-electron- and the ion-fluid, indexes 'b', 'e' and 'i' respectively, are assumed to obey the following particle conservation equations [3]:

$$\frac{d\rho_b u_b}{dx} = -n_0\sigma_{b0}L\rho_b u_b$$

$$\frac{d\rho_e u_e}{dx} = 2n_0\sigma_{b0}L\rho_b u_b$$

$$\frac{d\rho_i u_i}{dx} = n_0\sigma_{b0}L\rho_b u_b$$

Subscript '0' refers to the neutral background. The model assumes that beam electrons experience only ionizing collision with the background. The result of one such collision is modeled as the loss of one electron from the beam and the gain of two electrons and one ion for the plasma.

2.2 Momentum and energy equations, outer region

In the outer region an overall charge neutrality is assumed, together with a negligible beam density ρ_b compared to the bulk densities so that $\rho_i = \rho_e = \rho$. Due to its high kinetic energy, the beam may be considered cold and the conservation of beam momentum reduces to

$$u_b\frac{du_b}{dx} - \frac{m_i}{m_e}\frac{d\Phi}{dx} = 0\ .$$

The timescale of the bulk electron motion is infinitesimal compared with the timescale of motion of the heavy ions. They instantly adjust to their environment of ions. We therefore neglect the acceleration term and set their temperature to be constant. The momentum conservation of bulk electrons then states that

$$\frac{d\rho}{dx} - \rho\frac{d\Phi}{dx} = \frac{m_e}{m_i}(\Gamma_b f_{b0} - \Gamma_e f_{e0} - (\Gamma_e - \Gamma_i)f_{ei}).$$

$\Gamma_\alpha = \rho_\alpha u_\alpha$, $\alpha =' b', ' e'$ and $' i'$ are the particle fluxes from the particle conservation equations. The heavy ions are accelerated, and are generally assumed to have a temperature gradient,

$$\rho u_i\frac{du_i}{dx} + \tau_i\frac{d\rho}{dx} + \rho\frac{d\Phi}{dx} + \rho\frac{d\tau_i}{dx}$$

$$= -u_i n_0\sigma_{b0}L\Gamma_b - \Gamma_i f_{i0} - (\Gamma_i - \Gamma_e)f_{ie}.$$

Without an ion temperature gradient in this model it can be shown that the ion-streaming towards the cathode will not pass through the sonic point. To demonstrate the transition from sub-to supersonic streaming of ions we here for simplicity use the temperature gradient from classical heat conduction, and hence use

$$\tau_i^{5/2}\frac{d\tau_i}{dx} = c_T,$$

where c_T is a constant determined by the (ion) temperature difference between the cathode and the anode. With this equation we have a closed set of equations for u_i, ρ, Φ and τ_i. The set of equations may be put on the general form

$$A(\mathbf{y})\frac{d\mathbf{y}}{dx} = \mathbf{b}(x,\mathbf{y})$$

where now $A(\mathbf{y}) = \begin{pmatrix} \rho & u_i & 0 & 0 \\ 0 & 1 & -\rho & 0 \\ \rho u_i & \tau_i & \rho & \rho \\ 0 & 0 & 0 & \tau_i^{5/2} \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} u_i \\ \rho \\ \Phi \\ \tau_i \end{pmatrix}$

and the components of **b** are given by the right hand sides of the equations for particle conservation of ions, momentum conservation of bulk electrons and ions, and the ion temperature equation, respectively.

From Cramer's rule, the gradients are

$$\frac{dy_i}{dx} = \frac{N_i(x,\mathbf{y})}{\Delta(\mathbf{y})},\ i = 1,\ldots,4,$$

where $\Delta(\mathbf{y})$ is the determinant of **A** and $N_i(x,\mathbf{y})$ are similar determinants except that vector **b** replaces the i'th column. The system of equations is singular when $\Delta(\mathbf{y}) = -\rho^2(u_i^2 - u_{ts}^2)\tau_i^{5/2}$, where $u_{ts} = \sqrt{1+\tau_i}$, equals zero, i.e. at the sonic velocity. For solutions to pass through this velocity also all N_i must be zero there. The equations may be transformed into an autonomous form of 5 equations of 5 unknowns $(x,\mathbf{y})$ by introducing a new independent variable *t*, setting

$$\frac{dx}{dt} = \Delta(\mathbf{y}),\ \frac{dy_i}{dt} = \frac{dy_i}{dx}\frac{dx}{dt} = N_i(x,\mathbf{y}),\ i = 1..4.$$

A closer study [2] of solutions of these 5 equations in the neighborhood of the above singularity, which is an equilibrium in the $(x,\mathbf{y})$-space, reveals the equilibrium is a saddle point in a two-dimensional subset of the 5 dimensional $(x,\mathbf{y})$-space. Two solutions pass exactly through the equilibrium, one from sub- to supersonic velocity, and one from super- to subsonic velocity.

2.3 Equations, inner region (cathode sheath)

The non-dimensional variables are the same, except that the characteristic lengthscale is changed from the length of the fieldlines L to the Debye-length λ_D, the lengthscale of the sheath. The mean free path of the particles in the model is much larger than λ_D. Hence one may assume that no collisions take place here. The particle flow is cold. In this regime all the unknowns are functions of the electrical potential Φ only, and the problem reduces to solving one implicit integral equation in Φ as a function of the new independent variable $\xi = \frac{x}{\lambda_D}$.

3. Solutions

In the cathode sheath the solutions should fulfill the boundary conditions $u_b = u_{bc}$, $\rho_b = \rho_{bc}$, $\Phi = \Phi_c$ at $\xi = 0$. Subscript 'c' refers to cathode values, assumed known quantities. As $\xi \to \infty$, these solutions should match solutions at *x*=0 from the outer region. Outer region solutions must be properly tuned and meet conditions also at the anode.

Background gas was assumed to be Hydrogen. Values in the numerical experiment were set to $n_0 = 10^{18} m^{-3}$, $n_{pl} = n_{bc} = 10^{17} m^{-3}$, $v_{bc} = 3 \cdot 10^5 m/s$, $L = 40m$, $\Phi_A - \Phi_c = 100V$, $T_e = 10eV$, $T_A = 1eV$. Subscript 'A' denotes anode. The collision cross sections used are $\sigma_{b0} = 0.53 \cdot 10^{-20} m^2$, $\sigma_{e0} = 9.5 \cdot 10^{-20} m^2$ and $\sigma_{i0} = 20.0 \cdot 10^{-20} m^2$, and a classical electron-ion collision frequency was assumed. Fig.1 shows the ion flow in the outer region, while Fig.2 shows the transition from sub- to supersonic flow. The transition takes place at $s_s = 1.99cm \approx 267\lambda_D$ rather close to the cathode sheath in the outer region. The sheath extends approximately $10\lambda_D$ from the cathode. The smooth curves are the physically acceptable ones. A non-physical (dashed) branch of flow is also shown.

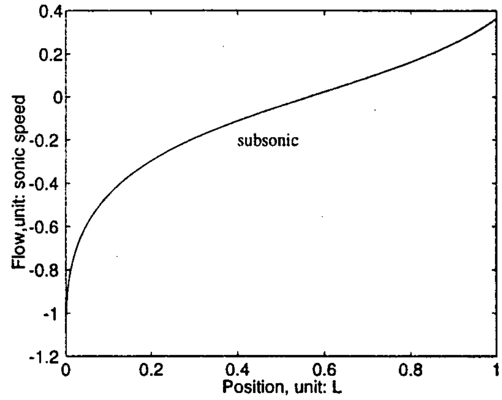

Fig.1. Ion flow in outer region. Cathode at left.

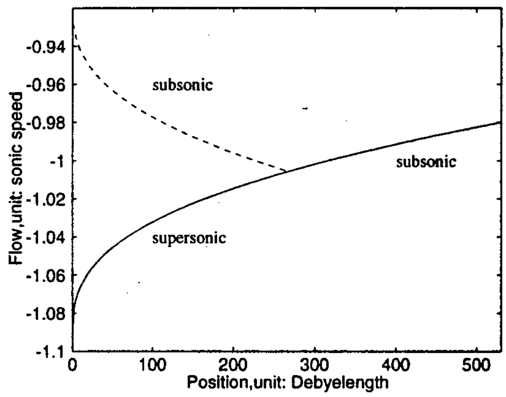

Fig.2. Ion flow close to the cathode sheath. Cathode at left.

4. References

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[2] Z. Bilicki, C.Dafermos, J. Kestin, G. Majda, D.L. Zeng: Int. J. Multiphase Flow, **13** (1987) 511.

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