The interplay between graphs, simplicial complexes and commutative algebra
Informatics seminar, April 2016

Gunnar Fløystad

April 8, 2016

Simplicial complexes

V finite set.

Definition

A simplicial complex Δ on V is a family of subset of V such that if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$.

Example

$\{1,2,3\}, \{3,4\}$,
$\{1,2\}, \{2,3\}, \{1,3\}$,
$\{1\}, \{2\}, \{3\}, \{4\}$,
$\emptyset$

Topological realization:

Triangle with an edge attached.

Squarefree monomial ideals

Example

$x_2x_4x_5x_7$ is a squarefree monomial. Write it $m_{\{2,4,5,7\}}$.

Definition

An ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is a *squarefree monomial ideal* if it is generated by squarefree monomials.

Stanley-Reisner ring

Simiplicial complexes on $[n] = \{1, 2, 3, \ldots, n\} \overset{1-1}{\leftrightarrow}$ squarefree monomial ideals I_Δ:

$$R \notin \Delta \quad \Leftrightarrow \quad m_R \in I_\Delta.$$

Stanley-Reisner ring

Simiplicial complexes on $[n] = \{1, 2, 3, \ldots, n\} \overset{1-1}{\Leftrightarrow}$ squarefree monomial ideals I_Δ:

$$R \notin \Delta \quad \Leftrightarrow \quad m_R \in I_\Delta.$$

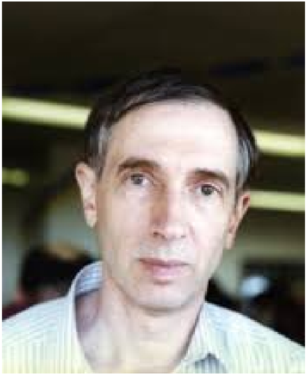
Definition

Stanley-Reisner ring $\Bbbk[\Delta] = \Bbbk[x_1, \ldots, x_n]/I_\Delta$.

Note: $R \in \Delta \Leftrightarrow m_R$ is nonzero in $\Bbbk[\Delta]$.

Founding fathers

Richard Stanley, MIT



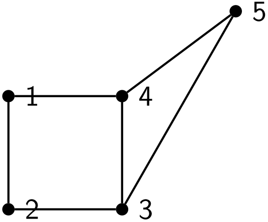
Proof of upper bound conjecture for simplicial spheres, 1975.

Melvin Hochster, U. of Michigan

Seminal paper, 1975

Rings from graphs I
Stanley-Reisner ring

Graph G:



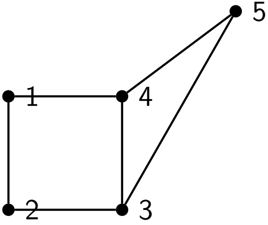
Stanley-Reisner ideal:

$$I_G = \langle x_1x_3, x_2x_4, x_1x_5, x_2x_5, x_3x_4x_5 \rangle.$$

Rings from graphs I

Stanley-Reisner ring

Graph G:



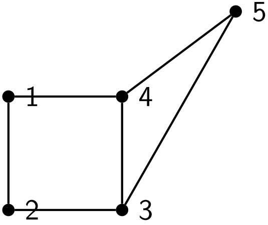
Stanley-Reisner ideal:

$$I_G = \langle x_1x_3, x_2x_4, x_1x_5, x_2x_5, x_3x_4x_5 \rangle.$$

Stanley-Reisner ring $\mathbb{k}[G] = \mathbb{k}[x_1, \ldots, x_5]/I_G$.

Rings from graphs II

Edge ideals

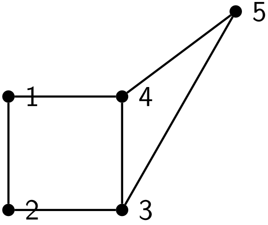


Edge ideal:

$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5, x_3x_5 \rangle.$$

Rings from graphs II

Edge ideals



Edge ideal:

$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5, x_3x_5 \rangle.$$

The ring $\Bbbk[x_1, \ldots, x_5]/I(G)$ is also a Stanley-Reisner ring $\Bbbk[\Delta]$. The simplicial complex Δ is the *independence complex* of the graph G.

Algebra

Let Δ be a simplicial complex.

$\mathbb{k}[\Delta]$ is a ring. Use machinery of algebra to study Δ.

Bases and minimal generating sets

Field $\mathbb{k} \rightsquigarrow$ vector space $V \rightsquigarrow$ dimension of V.

Ring $R \rightsquigarrow$ module $M \rightsquigarrow$?

Modules rarely have a basis. Instead we take a *minimal* generating set.

Example I

Example

Ring $R = \Bbbk[x, y, z]$. Consider the ideal $I = (xz, yz) \subseteq R$ (this is a module).

- xz and yz form a minimal generating set.
- Dependency $y \cdot xz - x \cdot yz = 0$.

Example I

Example

Ring $R = \Bbbk[x, y, z]$. Consider the ideal $I = (xz, yz) \subseteq R$ (this is a module).

- xz and yz form a minimal generating set.
- Dependency $y \cdot xz - x \cdot yz = 0$.
- Make a map

$$\begin{aligned} I &\xleftarrow{d} Re \oplus Rf = R^2 \\ xz &\leftarrow \quad e \\ yz &\leftarrow \quad f \end{aligned}$$

- Make a map

$$
\begin{aligned}
I \xleftarrow{d} & \; Re \oplus Rf = R^2 \\
xz \leftarrow & \quad e \\
yz \leftarrow & \quad f
\end{aligned}
$$

- We see that $(y, -x) = ye - xz$ is in the *kernel* (nullspace) of d, in fact generates the kernel.

- Make a map

$$I \xleftarrow{d} Re \oplus Rf = R^2$$
$$xz \leftarrow \quad e$$
$$yz \leftarrow \quad f$$

- We see that $(y, -x) = ye - xz$ is in the *kernel* (nullspace) of d, in fact generates the kernel.
- The *kernel* K of d are the $(s, t) \in R^2$ such that $(s, t) \xrightarrow{d} 0$. The kernel K is a submodule of R^2.

- Make a map

$$\begin{aligned} I \xleftarrow{d} & \; Re \oplus Rf = R^2 \\ xz \leftarrow & \; e \\ yz \leftarrow & \; f \end{aligned}$$

- We see that $(y, -x) = ye - xz$ is in the *kernel* (nullspace) of d, in fact generates the kernel.
- The *kernel* K of d are the $(s,t) \in R^2$ such that $(s,t) \xrightarrow{d} 0$. The kernel K is a submodule of R^2.
- Here $(y, -x)$ is in fact a *basis* for the kernel K, meaning that the map

$$\begin{aligned} K \leftarrow & \; R^1 = Rg \\ (y, -x) \leftarrow & \; g \\ ye - xf \leftarrow & \; g \end{aligned}$$

is an isomorphism.

Continuing

In general:

- Iterate process. Find minimal generating set $u_1, \ldots, u_p$ of K. Make a map $K \xleftarrow{d_2} R^p$.
- Continue and get

$$I \leftarrow R^r \xleftarrow{d_1(=d)} R^p \xleftarrow{d_2} R^q \leftarrow \cdots ,$$

called the *resolution* of I.

Example II

Example

$$I = (xz, yz) \leftarrow R^2 \leftarrow R^1.$$

Example II

Example

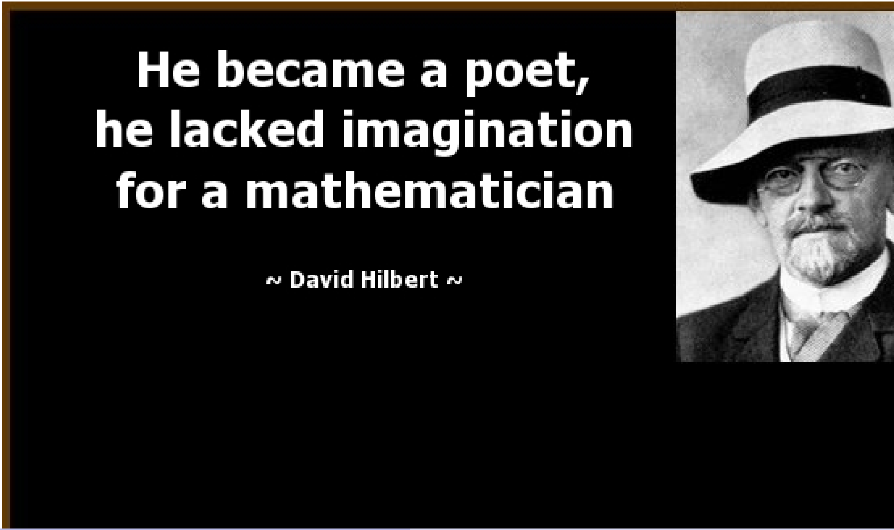
$$I = (xz, yz) \leftarrow R^2 \leftarrow R^1.$$

Variations:

$$R/I \leftarrow R^1 \xleftarrow{[xz,yz]} R^2 \xleftarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R^1$$

$$I \leftarrow R(-2)^2 \leftarrow R(-3)^1$$

Founder commutative algebra
1862-1943



Commutative Algebra

Founding theorems

Theorem (1890)

Any ideal in a polynomial ring has a finite generating set.

Commutative Algebra
Founding theorems

Theorem (1890)

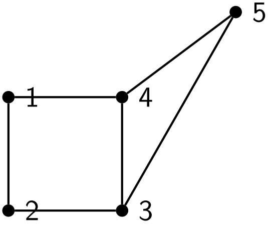
Any ideal in a polynomial ring has a finite generating set.

Theorem (1890)

Any finitely generated module over a polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$ *has a minimal free resolution of length* $\leq n$.

Rings from graphs

Graph G:



Stanley-Reisner ideal:

$$I_G = \langle x_1x_3, x_2x_4, x_1x_5, x_2x_5, x_3x_4x_5 \rangle.$$

Stanley-Reisner ring $\Bbbk[G] = \Bbbk[x_1, \ldots, x_5]/I_G$.

Resolution of I_G

Resolution

$$R/I_G \leftarrow R \leftarrow \begin{matrix} R(-2)^4 \\ \oplus \\ R(-3)^1 \end{matrix} \leftarrow \begin{matrix} R(-3)^3 \\ \oplus \\ R(-4)^3 \end{matrix} \leftarrow R(-5)^2.$$

Resolution of I_G

Resolution

$$R/I_G \leftarrow R \leftarrow \begin{matrix} R(-2)^4 \\ \oplus \\ R(-3)^1 \end{matrix} \leftarrow \begin{matrix} R(-3)^3 \\ \oplus \\ R(-4)^3 \end{matrix} \leftarrow R(-5)^2.$$

```
             0 1 2 3
o4 = total: 1 5 6 2
          0: 1 . . .
          1: . 4 3 .
          2: . 1 3 2
```

Resolution of I_G

Resolution

$$R/I_G \leftarrow R \leftarrow \begin{array}{c} R(-2)^4 \\ \oplus \\ R(-3)^1 \end{array} \leftarrow \begin{array}{c} R(-3)^3 \\ \oplus \\ R(-4)^3 \end{array} \leftarrow R(-5)^2.$$

```
              0 1 2 3
o4 = total: 1 5 6 2
          0: 1 . . .
          1: . 4 3 .
          2: . 1 3 2
```

- Dimension of G: 1.
- Length of resolution: 3
- Number of vertices: 5.

Cohen-Macaulay simplicial complexes

Simplicial complex Δ. Always:

$$\text{length resolution } \geq (n-1) - \text{ dimension } \Delta.$$

Definition

Δ is *Cohen-Macaulay* if we have equality above.

Gorenstein simplicial complexes

Stanley-Reisner ideal of six-cycle:

$$I_{C_6} = (x_1x_3, x_2x_4, x_3x_5, x_4x_6, x_5x_1, x_6x_2, x_1x_4, x_2x_5, x_3x_6).$$

Betti diagram of resolution:

```
            0 1  2 3 4
o18 = total: 1 9 16 9 1
          0: 1 .  . . .
          1: . 9 16 9 .
          2: . .  . . 1
```

Gorenstein simplicial complexes

Stanley-Reisner ideal of six-cycle:

$$I_{C_6} = (x_1x_3, x_2x_4, x_3x_5, x_4x_6, x_5x_1, x_6x_2, x_1x_4, x_2x_5, x_3x_6).$$

Betti diagram of resolution:

```
                 0 1  2 3 4
o18 = total: 1 9 16 9 1
          0: 1 .  . . .
          1: . 9 16 9 .
          2: . .  . . 1
```

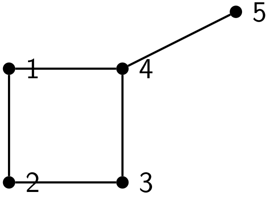
Definition

Δ is *Gorenstein* if:

- Δ is Cohen-Macaulay
- Resolution of Δ has symmetric form

Resolution of edge ideal

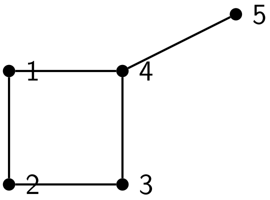
Graph G:



$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5 \rangle$$

Resolution of edge ideal

Graph G:

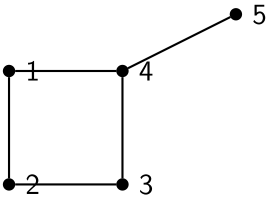


$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5 \rangle$$

```
             0 1 2 3
o13 = total: 1 5 6 2
          0: 1 . . .
          1: . 5 6 2
```

Resolution of edge ideal

Graph G:

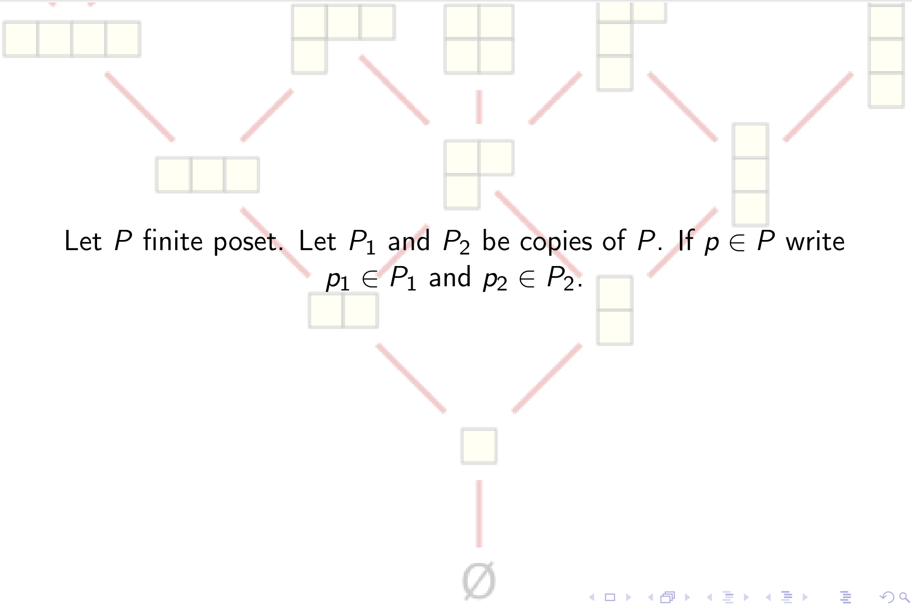


$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5 \rangle$$

```
                0 1 2 3
o13 = total: 1 5 6 2
           0: 1 . . .
           1: . 5 6 2
```

- Dimension independence complex: 2
- Length resolution: 3
- Vertices: 5

Not Cohen-Macaulay!



Let P finite poset. Let P_1 and P_2 be copies of P. If $p \in P$ write $p_1 \in P_1$ and $p_2 \in P_2$.

Make a bipartite graph $G = (P_1, P_2)$ where the edges are pairs $\{p_1, q_2\}$ such that $p \leq q$.

Make a bipartite graph $G = (P_1, P_2)$ where the edges are pairs $\{p_1, q_2\}$ such that $p \leq q$.

Theorem (J.Herzog, T.Hibi, 2004)

A bipartite graph is Cohen-Macaulay if and only if it comes from a poset as above.

Generalization

Consider a chain $p^1 \leq p^2 \leq \cdots \leq p^n$ in the poset P.

Squarefree monomial: $x_{1,p^1} x_{2,p^2} \cdots x_{n,p^n}$.
$L(n, P)$: ideal generated by all such monomials.

Generalization

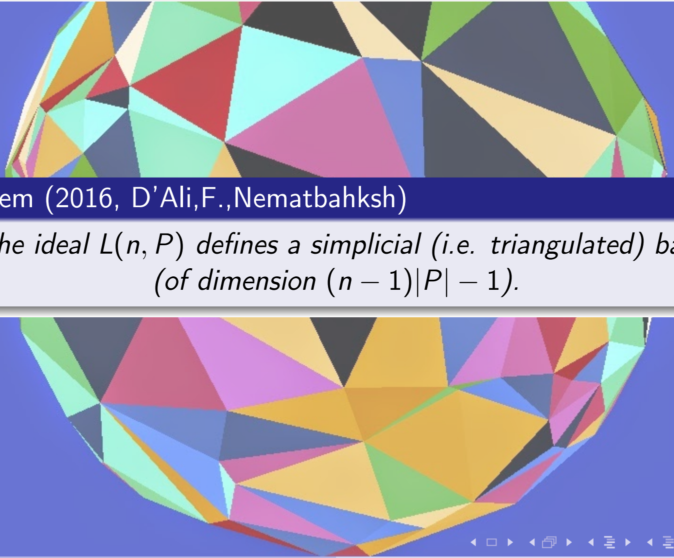
Consider a chain $p^1 \leq p^2 \leq \cdots \leq p^n$ in the poset P.

Squarefree monomial: $x_{1,p^1} x_{2,p^2} \cdots x_{n,p^n}$.
$L(n, P)$: ideal generated by all such monomials.

Theorem (2011, Herzog et.al.)

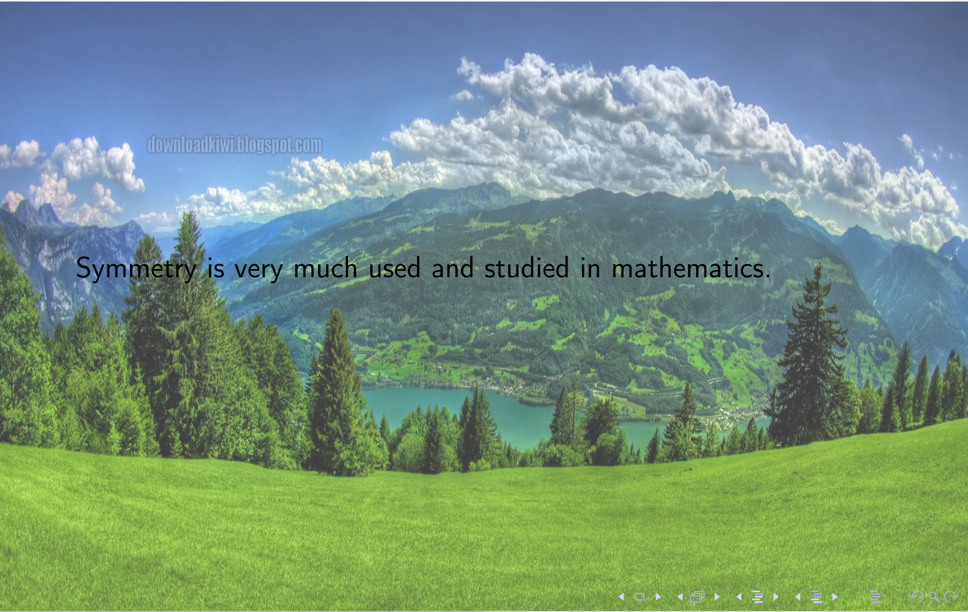
The ideal $L(n, P)$ defines a Cohen-Macaulay simplicial complex of codimension $|P|$ in the simplex on $n|P|$ vertices.

Balls and spheres

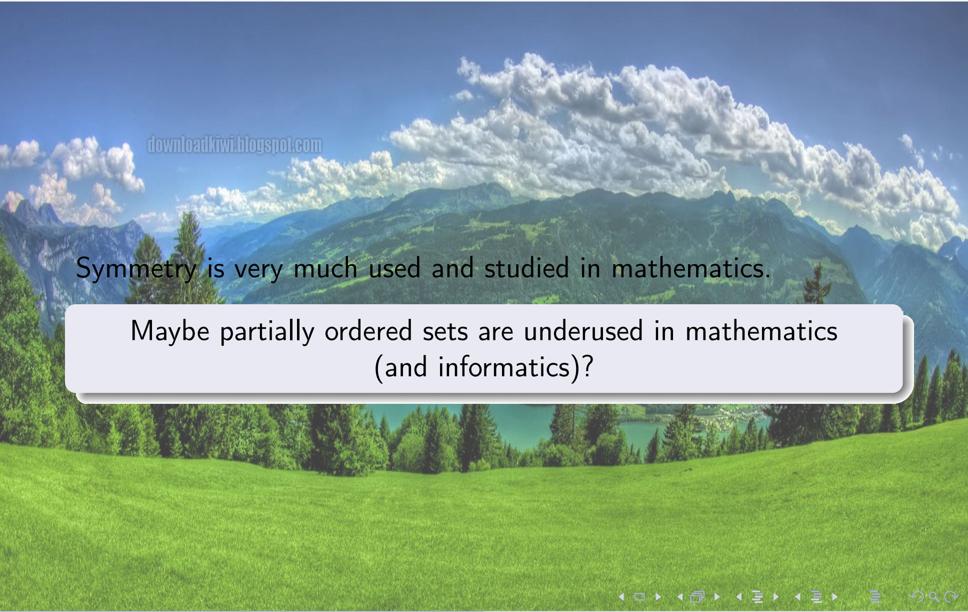


Theorem (2016, D'Ali,F.,Nematbahksh)

The ideal $L(n, P)$ defines a simplicial (i.e. triangulated) ball (of dimension $(n-1)|P|-1$).



Symmetry is very much used and studied in mathematics.



Symmetry is very much used and studied in mathematics.

Maybe partially ordered sets are underused in mathematics (and informatics)?