Starting with Maxwell's equations in frequency domain with PML additions denoted by $\bar{s}$, where '$\overline{\ }$' denotes a matrix.

$$\nabla \times E(\omega) = -j\omega\mu_0\overline{\mu_r}\bar{s}H(\omega)$$

$$\nabla \times H(\omega) = \bar{\sigma}E(\omega) + j\omega\bar{s}D(\omega)$$

$$D(\omega) = \epsilon_0 \cdot \overline{\epsilon_r}(\omega) \cdot E(\omega)$$

$$\bar{s} = \begin{bmatrix} S_y & 0 & 0 \\ 0 & \frac{1}{S_y} & 0 \\ 0 & 0 & S_y \end{bmatrix}, \qquad S_y = 1 + \frac{\sigma_y'(y)}{j\omega\epsilon_0}, \qquad \sigma_y' = \frac{\epsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3$$

$$\overline{\mu_r} = \begin{bmatrix} \mu_{rx} & 0 & 0 \\ 0 & \mu_{ry} & 0 \\ 0 & 0 & \mu_{rz} \end{bmatrix}, \qquad \overline{\epsilon_r} = \begin{bmatrix} \epsilon_{rx} & 0 & 0 \\ 0 & \epsilon_{ry} & 0 \\ 0 & 0 & \epsilon_{rz} \end{bmatrix}, \qquad \bar{\sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

where L_y, is the length of the PML in the direction of y. Waves propagating in the y-direction will then be absorbed. Note that this conductivity σ' is only diffesrent from the value of 0 when inside the PML, and it is also assumed a diagonally anisotropic material. Then the normalization of the electric field is done in the following way,

$$\tilde{E} = \sqrt{\frac{\epsilon_0}{\mu_0}}E = \frac{1}{\eta_0}E$$

$$\widetilde{D} = \sqrt{\frac{1}{\epsilon_0\mu_0}}D = c_0 D$$

Where $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\epsilon_0\mu_0} \cdot \frac{1}{\epsilon_0} = \frac{1}{c_0\epsilon_0}$. Substituting the normalization into Maxwell equations,

$$\nabla \times \tilde{E}(\omega) = -\left(\frac{1}{\eta_0}\right) \cdot j\omega\mu_0\overline{\mu_r}\bar{s}H(\omega)$$

$$\nabla \times H(\omega) = \left(\frac{1}{\eta_0}\right)^{-1} \sigma\tilde{E}(\omega) + (c_0)^{-1}j\omega\bar{s}D(\omega)$$

$$\widetilde{D}(\omega) = \left(\sqrt{\frac{1}{\epsilon_0\mu_0}}\right)\left(\sqrt{\frac{\epsilon_0}{\mu_0}}\right)^{-1} \epsilon_0 \cdot \overline{\epsilon_r}\tilde{E}(\omega)$$

Simplifying the equations,

$$\nabla \times \tilde{E}(\omega) = -j\omega \cdot \frac{1}{c_0} \cdot \overline{\mu_r}\bar{s}H(\omega)$$

$$\nabla \times H(\omega) = \eta_0\sigma \cdot \tilde{E}(\omega) + j\omega \cdot \frac{1}{c_0} \cdot \bar{s}\widetilde{D}(\omega)$$

$$\widetilde{D}(\omega) = \overline{\epsilon_r}\tilde{E}(\omega)$$

Expanding these equations for each spatial coordinate

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega \cdot \left[\frac{\mu_{rx}}{c_0} \cdot S_y \cdot H_x\right]$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega \cdot \left[\frac{\mu_{ry}}{c_0} \cdot \frac{1}{S_y} \cdot H_y\right]$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \cdot \left[\frac{\mu_{rz}}{c_0} \cdot S_y \cdot H_z\right]$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\eta_0 \sigma_x) \cdot \tilde{E}_x + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \widetilde{D}_x\right]$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \left(\eta_0 \sigma_y\right) \cdot \tilde{E}_y + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \frac{1}{S_y} \cdot \widetilde{D}_y\right]$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\eta_0 \sigma_z) \cdot \tilde{E}_z + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \widetilde{D}_z\right]$$

To reduce the dimensions, we consider an infinite rectangular beam. The cross-section will then be enough to describe the behavior of the electric and magnetic fields. For such a rectangular beam with infinite extent in the z-direction, changing the position in space along the z-direction will not change the fields, and thus all $\partial/\partial z = 0$.

$$\frac{\partial E_z}{\partial y} = -j\omega \cdot \left[\frac{\mu_{rx}}{c_0} \cdot S_y \cdot H_x\right]$$

$$-\frac{\partial E_z}{\partial x} = -j\omega \cdot \left[\frac{\mu_{ry}}{c_0} \cdot \frac{1}{S_y} \cdot H_y\right]$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \cdot \left[\frac{\mu_{rz}}{c_0} \cdot S_y \cdot H_z\right]$$

$$\frac{\partial H_z}{\partial y} = (\eta_0 \sigma_x) \cdot \tilde{E}_x + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \widetilde{D}_x\right]$$

$$-\frac{\partial H_z}{\partial x} = \left(\eta_0 \sigma_y\right) \cdot \tilde{E}_y + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \frac{1}{S_y} \cdot \widetilde{D}_y\right]$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\eta_0 \sigma_z) \cdot \tilde{E}_z + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \widetilde{D}_z\right]$$

Taking a closer at the reduced system, we uncover two independent systems of equations. These two systems form the transverse electric mode and the transverse magnetic mode.

$$\frac{\partial E_z}{\partial y} = -j\omega \cdot \left[\frac{\mu_{rx}}{c_0} \cdot S_y \cdot H_x\right]$$

$$-\frac{\partial E_z}{\partial x} = -j\omega \cdot \left[\frac{\mu_{ry}}{c_0} \cdot \frac{1}{S_y} \cdot H_y\right]$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\eta_0 \sigma_z) \cdot \tilde{E}_z + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \widetilde{D}_z\right]$$

And,

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \cdot \left[\left(\frac{\mu_{rz}}{c_0}\right) \cdot S_y \cdot H_z\right]$$

$$\frac{\partial H_z}{\partial y} = (\eta_0 \sigma_x) \cdot \tilde{E}_x + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \tilde{D}_x\right]$$

$$-\frac{\partial H_z}{\partial x} = (\eta_0 \sigma_y) \cdot \tilde{E}_y + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \frac{1}{S_y} \cdot \tilde{D}_y\right]$$

Both are needed for a complete picture, however, often only one mode is considered as it can explain the system given the source is polarized the correct way. We are more interested in investigating the electric field and using a simple model, and thus we only consider the transverse electric mode (i.e. only considering E_x, E_y, H_z). The following are the governing equations,

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = j\omega \cdot \left[\left(-\frac{\mu_{ry}}{c_0}\right) \cdot S_y \cdot H_z(\omega)\right]$$

$$\frac{\partial H_z}{\partial y} = (\eta_0 \sigma_x) \cdot \tilde{E}_x(\omega) + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot S_y \cdot \tilde{D}_x(\omega)\right]$$

$$-\frac{\partial H_z}{\partial x} = (\eta_0 \sigma_y) \cdot \tilde{E}_y(\omega) + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \frac{1}{S_y} \cdot \tilde{D}_y(\omega)\right]$$

$$\tilde{D}_x(\omega) = \epsilon_{rx} \tilde{E}_x(\omega)$$

$$\tilde{D}_y(\omega) = \epsilon_{ry} \tilde{E}_y(\omega)$$

Before we move the equations from the frequency domain to the time domain, we expand the S_y term representing the PML (absorbing layer) found in the curl equations,

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = j\omega \cdot \left[\left(-\frac{\mu_{ry}}{c_0}\right) \cdot \left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right) \cdot H_z\right]$$

$$\frac{\partial H_z}{\partial y} = (\eta_0 \sigma_x) \cdot \tilde{E}_x + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right) \cdot \tilde{D}_x\right]$$

$$-\frac{\partial H_z}{\partial x} = (\eta_0 \sigma_y) \cdot \tilde{E}_y + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \left(\frac{1}{1 + \frac{\sigma_y'}{j\omega\epsilon_0}}\right) \cdot \tilde{D}_y\right]$$

Multiplying with $\left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right)$ in the last curl equation,

$$\left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right)\left(-\frac{\partial H_z}{\partial x}\right) = [\eta_0 \sigma_y]\left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right) \cdot \tilde{E}_y + j\omega \cdot \left(\left[\frac{1}{c_0}\right] \cdot \tilde{D}_y\right)$$

Thus, the curl equations become,

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = j\omega \cdot \left[\left(-\frac{\mu_{ry}}{c_0}\right) \cdot \left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right) \cdot H_z\right]$$

$$\frac{\partial H_z}{\partial y} = (\eta_0 \sigma_x) \cdot \tilde{E}_x + j\omega \cdot \left[\left(\frac{1}{c_0}\right) \cdot \left(1 + \frac{\sigma_y'}{j\omega\epsilon_0}\right) \cdot \tilde{D}_x\right]$$

$$\left(1+\frac{\sigma_y'}{j\omega\epsilon_0}\right)\left(-\frac{\partial H_z}{\partial x}\right) = (\eta_0\sigma_y)\left(1+\frac{\sigma_y'}{j\omega\epsilon_0}\right)\cdot\tilde{E}_y + j\omega\cdot\left[\left(\frac{1}{c_0}\right)\cdot\tilde{D}_y\right]$$

Let the curl be denoted by C_z^E, C_x^H, C_y^H. Separating the operators $\frac{1}{j\omega}$ and $j\omega$ of frequency space,

$$C_z^E = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = j\omega\cdot\left[\left(-\frac{\mu_{ry}}{c_0} - \frac{1}{j\omega}\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)\cdot H_z\right]$$

$$C_x^H = \frac{\partial H_z}{\partial y} = (\eta_0\sigma_x)\tilde{E}_x + j\omega\cdot\left[\left(\frac{1}{c_0} + \frac{1}{j\omega}\frac{\sigma_y'}{c_0\epsilon_0}\right)\cdot\tilde{D}_x\right]$$

$$\left(1+\frac{1}{j\omega}\frac{\sigma_y'}{\epsilon_0}\right)C_y^H = \left(1+\frac{1}{j\omega}\frac{\sigma_y'}{\epsilon_0}\right)\left(-\frac{\partial H_z}{\partial x}\right) = \left(\eta_0\sigma_y + \frac{1}{j\omega}\frac{\eta_0\sigma_y\sigma_y'}{\epsilon_0}\right)\cdot\tilde{E}_y + j\omega\cdot\left[\left(\frac{1}{c_0}\right)\cdot\tilde{D}_y\right]$$

Expanding the brackets,

$$C_z^E = -j\omega\cdot\left[\left(\frac{\mu_{ry}}{c_0}\right)H_z\right] - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)H_z$$

$$C_x^H = (\eta_0\sigma_x)\tilde{E}_x + j\omega\cdot\left[\left(\frac{1}{c_0}\right)\tilde{D}_x\right] + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x$$

$$C_y^H + \frac{1}{j\omega}\left[\left(\frac{\sigma_y'}{\epsilon_0}\right)C_y^H\right] = (\eta_0\sigma_y)\tilde{E}_y + \frac{1}{j\omega}\left[\left(\frac{\eta_0\sigma_y\sigma_y'}{\epsilon_0}\right)\tilde{E}_y\right] + j\omega\cdot\left[\left(\frac{1}{c_0}\right)\tilde{D}_y\right]$$

Moving the equations from frequency domain to the time domain. The conversion follows the following rules: $j\omega[\quad]\rightarrow\frac{\partial}{\partial t}[\quad]$, and $\frac{1}{j\omega}[\quad]\rightarrow\int_0^t[\quad]\,dt$. We assume that the constants in front of the field values does not change with time.

$$C_z^E|_t = -\left(\frac{\mu_{ry}}{c_0}\right)\frac{\partial H_z}{\partial t}\bigg|_t - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)H_z|_t$$

$$C_x^H|_t = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{c_0}\right)\frac{\partial\tilde{D}_x}{\partial t}\bigg|_t + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$C_y^H\big|_t + \left(\frac{\sigma_y'}{\epsilon_0}\right)\int_0^t C_y^H dt = (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'}{\epsilon_0}\right)\int_0^t \tilde{E}_y dt + \left(\frac{1}{c_0}\right)\frac{\partial\tilde{D}_y}{\partial t}\bigg|_t$$

Then we discretize the curl equations. The electric field is both temporally and spatially displaced in regard to the magnetic field. This automatically conserves the divergence relationships, and the electromagnetic boundary conditions. The electric field is given on integer steps (in space and time), meanwhile the magnetic field is given on half integer steps (in space and time). Before we implement the spatial displacements we first consider the displaced time.

Maxwell's equation is only given for the same point in time and space, so to account for the temporal and spatial displacement, an approximation has to be done, the terms written in red is not defined for the desired time step, and has to be defined afterwards by approximation.

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} - H_z\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)H_z\big|_t$$

$$\mathcal{C}_x^H\big|_t = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{c_0}\right)\frac{\tilde{D}_x\big|_{t+\frac{\Delta t}{2}} - \tilde{D}_x\big|_{t-\frac{\Delta t}{2}}}{\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\mathcal{C}_y^H\big|_t + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\mathcal{C}_y^H\big|_{t_i} = (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{c_0}\right)\frac{\tilde{D}_y\big|_{t+\frac{\Delta t}{2}} - \tilde{D}_y\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

The temporal approximations for the red terms are given below:

$$H_z\big|_t = \frac{H_z\big|_{t+\frac{\Delta t}{2}} + H_z\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\mathcal{C}_x^H\big|_t = \frac{\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\tilde{D}_x\big|_{t+\frac{\Delta t}{2}} - \tilde{D}_x\big|_{t-\frac{\Delta t}{2}} = \frac{\tilde{D}_x\big|_{t+\Delta t} + \tilde{D}_x\big|_t}{2} - \frac{\tilde{D}_x\big|_t + \tilde{D}_x\big|_{t-\Delta t}}{2} = \frac{\tilde{D}_x\big|_{t+\Delta t} - \tilde{D}_x\big|_{t-\Delta t}}{2}$$

$$\mathcal{C}_y^H\big|_t = \frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\sum_{t_i=0}^{t}\mathcal{C}_y^H\big|_{t_i} = \frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{4} + \sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i}$$

$$\tilde{D}_y\big|_{t+\frac{\Delta t}{2}} - \tilde{D}_y\big|_{t-\frac{\Delta t}{2}} = \frac{\tilde{D}_y\big|_{t+\Delta t} + \tilde{D}_y\big|_t}{2} - \frac{\tilde{D}_y\big|_t + \tilde{D}_y\big|_{t-\Delta t}}{2} = \frac{\tilde{D}_y\big|_{t+\Delta t} - \tilde{D}_y\big|_{t-\Delta t}}{2}$$

Using the approximations the discrete curl equations become,

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} - H_z\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} + H_z\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\frac{\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}}}{2} = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{c_0}\right)\frac{\frac{\tilde{D}_x\big|_{t+\Delta t} - \tilde{D}_x\big|_{t-\Delta t}}{2}}{\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{2} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\left(\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{4} + \sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i}\right)$$

$$= (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{c_0}\right)\frac{\frac{\tilde{D}_y\big|_{t+\Delta t} - \tilde{D}_y\big|_{t-\Delta t}}{2}}{\Delta t}$$

Cleaning up the fractions,

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} - H_z\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} + H_z\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\frac{\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}}}{2} = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{2c_0}\right)\frac{\tilde{D}_x\big|_{t+\Delta t} - \tilde{D}_x\big|_{t-\Delta t}}{\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{2} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\left(\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{4} + \sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} \mathcal{C}_y^H\big|_{t_i}\right)$$
$$= (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{2c_0}\right)\frac{\tilde{D}_y\big|_{t+\Delta t} - \tilde{D}_y\big|_{t-\Delta t}}{\Delta t}$$

Highlighting the time-step driving the evolution of the system,

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} - H_z\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \left(\frac{\mu_{ry}\sigma_y'}{c_0\epsilon_0}\right)\frac{H_z\big|_{t+\frac{\Delta t}{2}} + H_z\big|_{t-\frac{\Delta t}{2}}}{2}$$

$$\frac{\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}}}{2} = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{2c_0}\right)\frac{\tilde{D}_x\big|_{t+\Delta t} - \tilde{D}_x\big|_{t-\Delta t}}{\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{2} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\left(\frac{\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}}}{4} + \sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} \mathcal{C}_y^H\big|_{t_i}\right)$$
$$= (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{2c_0}\right)\frac{\tilde{D}_y\big|_{t+\Delta t} - \tilde{D}_y\big|_{t-\Delta t}}{\Delta t}$$

The next time step (marked in yellow) is then separated and written as a function of previous field values to form the update equations. Starting the process by expanding the fractions:

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0\Delta t}\right)H_z\big|_{t+\frac{\Delta t}{2}} + \left(\frac{\mu_{ry}}{c_0\Delta t}\right)H_z\big|_{t-\frac{\Delta t}{2}} - \left(\frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)H_z\big|_{t+\frac{\Delta t}{2}} - \left(\frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)H_z\big|_{t-\frac{\Delta t}{2}}$$

$$\left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}} = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t+\Delta t} - \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t-\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\left(\frac{1}{2}\right)\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2}\right)\mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}} + \left(\frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i}$$
$$= (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t+\Delta t} - \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t-\Delta t}$$

Factorizing,

$$\mathcal{C}_z^E\big|_t = -\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right) H_z\big|_{t+\frac{\Delta t}{2}} + \left(\frac{\mu_{ry}}{c_0\Delta t} - \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right) H_z\big|_{t-\frac{\Delta t}{2}}$$

$$\left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}} = (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t+\Delta t} - \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t-\Delta t} + \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t$$

$$\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i}$$
$$= (\eta_0\sigma_y)\tilde{E}_y\big|_t + \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t+\Delta t} - \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t-\Delta t}$$

Rearranging,

$$\mathcal{C}_z^E\big|_t - \left(\frac{\mu_{ry}}{c_0\Delta t} - \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right) H_z\big|_{t-\frac{\Delta t}{2}} = -\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right) H_z\big|_{t+\frac{\Delta t}{2}}$$

$$\left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2}\right)\mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}} - (\eta_0\sigma_x)\tilde{E}_x\big|_t + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t-\Delta t} - \left(\frac{\sigma_y'}{c_0\epsilon_0}\right)\tilde{D}_x\big|_t = \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_x\big|_{t+\Delta t}$$

$$\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)\mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}} + \left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i} - (\eta_0\sigma_y)\tilde{E}_y\big|_t$$
$$- \left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t-\Delta t} = \left(\frac{1}{2c_0\Delta t}\right)\tilde{D}_y\big|_{t+\Delta t}$$

Dividing by the factor,

$$\left[\frac{1}{-\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)}\right]\mathcal{C}_z^E\big|_t - \left[\frac{\left(\frac{\mu_{ry}}{c_0\Delta t} - \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)}{-\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)}\right] H_z\big|_{t-\frac{\Delta t}{2}} = H_z\big|_{t+\frac{\Delta t}{2}}$$

$$\left[\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\mathcal{C}_x^H\big|_{t+\frac{\Delta t}{2}} + \left[\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\mathcal{C}_x^H\big|_{t-\frac{\Delta t}{2}} - \left[\frac{(\eta_0\sigma_x)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\tilde{E}_x\big|_t + \left[\frac{\left(\frac{1}{2c_0\Delta t}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\tilde{D}_x\big|_{t-\Delta t} - \left[\frac{\left(\frac{\sigma_y'}{c_0\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\tilde{D}_x\big|_t$$
$$= \tilde{D}_x\big|_{t+\Delta t}$$

$$\left[\frac{\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\mathcal{C}_y^H\big|_{t+\frac{\Delta t}{2}} + \left[\frac{\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\mathcal{C}_y^H\big|_{t-\frac{\Delta t}{2}} + \left[\frac{\left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H\big|_{t_i} - \left[\frac{(\eta_0\sigma_y)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\tilde{E}_y\big|_t$$
$$- \left[\frac{\left(\frac{\eta_0\sigma_y\sigma_y'\Delta t}{\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\sum_{t_i=0}^{t}\tilde{E}_y\big|_{t_i} + \left[\frac{\left(\frac{1}{2c_0\Delta t}\right)}{\left(\frac{1}{2c_0\Delta t}\right)}\right]\tilde{D}_y\big|_{t-\Delta t} = \tilde{D}_y\big|_{t+\Delta t}$$

Re-expressing the coefficients (temporary coefficients, these will be redefined later. They are just used to simplify the current coefficients. Temporary coefficients are denoted by *):

$$[h_{z1}^{*}]\mathcal{C}_{z}^{E}|_{t} - [h_{z2}^{*}]H_{z}|_{t-\frac{\Delta t}{2}} = H_{z}|_{t+\frac{\Delta t}{2}}$$

$$[d_{x1}^{*}]\mathcal{C}_{x}^{H}|_{t+\frac{\Delta t}{2}} + [d_{x2}^{*}]\mathcal{C}_{x}^{H}|_{t-\frac{\Delta t}{2}} - [d_{x3}^{*}]\,\tilde{E}_{x}|_{t} + [d_{x4}^{*}]\widetilde{D}_{x}|_{t-\Delta t} - [d_{x5}^{*}]\widetilde{D}_{x}|_{t} = \widetilde{D}_{x}|_{t+\Delta t}$$

$$[d_{y1}^{*}]\mathcal{C}_{y}^{H}|_{t+\frac{\Delta t}{2}} + [d_{y2}^{*}]\mathcal{C}_{y}^{H}|_{t-\frac{\Delta t}{2}} + [d_{y3}^{*}]\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} \mathcal{C}_{y}^{H}|_{t_i} - [d_{y4}^{*}]\tilde{E}_{y}|_{t} - [d_{y5}^{*}]\sum_{t_i=0}^{t}\tilde{E}_{y}|_{t_i} + [d_{y6}^{*}]\widetilde{D}_{y}|_{t-\Delta t} = \widetilde{D}_{y}|_{t+\Delta t}$$

With the constants given by,

$$h_{z1}^{*} = \frac{1}{-\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)} = \frac{1}{-\left(\frac{2\mu_{ry}\epsilon_0}{2c_0\epsilon_0\Delta t} + \frac{\mu_{ry}\sigma_y'\Delta t}{2c_0\epsilon_0\Delta t}\right)} = -\frac{2c_0\epsilon_0\Delta t}{2\mu_{ry}\epsilon_0 + \mu_{ry}\sigma_y'\Delta t} \coloneqq -h_{z1}$$

$$h_{z2}^{*} = \frac{\left(\frac{\mu_{ry}}{c_0\Delta t} - \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)}{-\left(\frac{\mu_{ry}}{c_0\Delta t} + \frac{\mu_{ry}\sigma_y'}{2c_0\epsilon_0}\right)} = -\frac{\left(\frac{2\mu_{ry}\epsilon_0}{2c_0\epsilon_0\Delta t} - \frac{\mu_{ry}\sigma_y'\Delta t}{2c_0\epsilon_0\Delta t}\right)}{\left(\frac{2\mu_{ry}\epsilon_0}{2c_0\epsilon_0\Delta t} + \frac{\mu_{ry}\sigma_y'\Delta t}{2c_0\epsilon_0\Delta t}\right)} = -\frac{2\mu_{ry}\epsilon_0 - \mu_{ry}\sigma_y'\Delta t}{2\mu_{ry}\epsilon_0 + \mu_{ry}\sigma_y'\Delta t} \coloneqq -h_{z2}$$

$$d_{x1}^{*} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = c_0\Delta t \coloneqq d_{x1}, \qquad d_{x2}^{*} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = c_0\Delta t \coloneqq d_{x1},$$

$$d_{x3}^{*} = \frac{(\eta_0\sigma_x)}{\left(\frac{1}{2c_0\Delta t}\right)} = 2c_0\eta_0\sigma_x\Delta t \coloneqq d_{x2}$$

$$d_{x4}^{*} = \frac{\left(\frac{1}{2c_0\Delta t}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = 1, \qquad d_{x5}^{*} = \frac{\left(\frac{\sigma_y'}{c_0\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = \frac{2\sigma_y'\Delta t}{\epsilon_0} \coloneqq d_{x3}$$

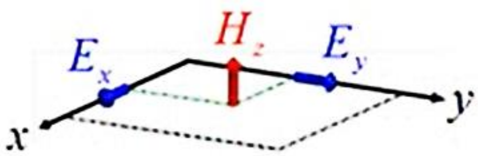
$$d_{y1}^{*} = \frac{\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = 2c_0\Delta t\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right) \coloneqq d_{y1},$$

$$d_{y2}^{*} = \frac{\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = 2c_0\Delta t\left(\frac{1}{2} + \frac{\sigma_y'\Delta t}{4\epsilon_0}\right) \coloneqq d_{y1}$$

$$d_{y3}^{*} = \frac{\left(\frac{\sigma_y'\Delta t}{\epsilon_0}\right)}{\left(\frac{1}{2c_0\Delta t}\right)} = 2\sigma_y'\eta_0(c_0\Delta t)^2 \coloneqq d_{y2}, \qquad d_{y4}^{*} = \frac{(\eta_0\sigma_y)}{\left(\frac{1}{2c_0\Delta t}\right)} = 2\sigma_y\eta_0c_0\Delta t \coloneqq d_{y3}$$

$$d_{y5}^* = \frac{\left(\frac{\eta_0 \sigma_y \sigma_y' \Delta t}{\epsilon_0}\right)}{\left(\frac{1}{2c_0 \Delta t}\right)} = 2\sigma_y \sigma_y' (\eta_0 c_0 \Delta t)^2 \coloneqq d_{y4}, \qquad d_{y6}^* = \frac{\left(\frac{1}{2c_0 \Delta t}\right)}{\left(\frac{1}{2c_0 \Delta t}\right)} = 1$$

The spatial coordinates $(i_x, i_y) \equiv (i_x \Delta x, i_y \Delta y)$ are then added to the system, along with the new constants. As previously the terms that are not defined on the given spatial position have to be approximated, and are colored red. Note that the $\epsilon_{rx}, \epsilon_{ry}, \sigma_x, \sigma_y, \mu_{ry}$ are not defined for the same point. For the spatial displacements (known as a Yee grid) we have the following illustration:



Thus,

$$H_z\left(i_x + \frac{1}{2}, i_y + \frac{1}{2}\right)\Big|_{t+\frac{\Delta t}{2}} = -[h_{z1}]\, C_z^E\left(i_x + \frac{1}{2}, i_y + \frac{1}{2}\right)\Big|_t + [h_{z2}]\, H_z\left(i_x + \frac{1}{2}, i_y + \frac{1}{2}\right)\Big|_{t-\frac{\Delta t}{2}}$$

$$\begin{aligned} \tilde{D}_x\left(i_x + \frac{1}{2}, i_y\right)\Big|_{t+\Delta t} &= [d_{x1}]\, C_x^H\left(i_x + \frac{1}{2}, i_y\right)\Big|_{t+\frac{\Delta t}{2}} + [d_{x1}]\, C_x^H\left(i_x + \frac{1}{2_x}, i_y\right)\Big|_{t-\frac{\Delta t}{2}} \\ &- [d_{x2}]\, \tilde{E}_x\left(i_x + \frac{1}{2}, i_y\right)\Big|_t + \tilde{D}_x\left(i_x + \frac{1}{2}, i_y\right)\Big|_{t-\Delta t} - [d_{x3}]\, \tilde{D}_x\left(i_x + \frac{1}{2}, i_y\right)\Big|_t \end{aligned}$$

$$\begin{aligned} \tilde{D}_y\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t+\Delta t} &= [d_{y1}]\, C_y^H\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t+\frac{\Delta t}{2}} + [d_{y1}]\, C_y^H\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t-\frac{\Delta t}{2}} \\ &+ [d_{y2}] \sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} C_y^H\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t_i} - [d_{y3}]\, \tilde{E}_y\left(i_x, i_y + \frac{1}{2}\right)\Big|_t \\ &- [d_{y4}] \sum_{t_i=0}^{t} \tilde{E}_y\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t_i} + \tilde{D}_y\left(i_x, i_y + \frac{1}{2}\right)\Big|_{t-\Delta t} \end{aligned}$$

In the matrices we use we cannot have half integer indexing, thus we have to adjust the curls forwards or backwards. The half indexing is then relegated to the differentiation of the curls. The indexing now represents the cell number and not the physical position,

$$H_z(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} = -[h_{z1}] C_z^E(i_x, i_y)^*\big|_t + [h_{z2}] H_z(i_x, i_y)\big|_{t-\frac{\Delta t}{2}}$$

$$\tilde{D}_x(i_x, i_y)\big|_{t+\Delta t} = [d_{x1}]C_x^H(i_x, i_y)^*\big|_{t+\frac{\Delta t}{2}} + [d_{x1}]C_x^H(i_{x_x}, i_y)^*\big|_{t-\frac{\Delta t}{2}} - [d_{x2}]\,\tilde{E}_x(i_x, i_y)\big|_t + \tilde{D}_x(i_x, i_y)\big|_{t-\Delta t} - [d_{x3}]\tilde{D}_x(i_x, i_y)\big|_t$$

$$\tilde{D}_y(i_x, i_y)\big|_{t+\Delta t} = [d_{y1}]C_y^H(i_x, i_y)^*\big|_{t+\frac{\Delta t}{2}} + [d_{y1}]C_y^H(i_x, i_y)^*\big|_{t-\frac{\Delta t}{2}} + [d_{y2}]\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}} C_y^H(i_x, i_y)^*\big|_{t_i} - [d_{y3}]\tilde{E}_y(i_x, i_y)\big|_t - [d_{y4}]\sum_{t_i=0}^{t}\tilde{E}_y(i_x, i_y)\big|_{t_i} + \tilde{D}_y(i_x, i_y)\big|_{t-\Delta t}$$

Since the electric field is defined on even integers its spatial derivatives have to be skewed forward away from the origin to compensate, and so that they are centered in space on top of the magnetic field which is defined on half integers. Meanwhile since the magnetic field is defined on half integers they are displaced outwards from the origin, skewing the spatial derivatives backwards one step will then center it on the position of the electric field. This means that the curls are written as,

$$C_z^E(i_x, i_y)^* = \frac{\tilde{E}_y(i_x+1, i_y) - \tilde{E}_y(i_x, i_y)}{\Delta x} - \frac{\tilde{E}_x(i_x, i_y+1) - \tilde{E}_x(i_x, i_y)}{\Delta y}$$

$$C_x^H(i_x, i_y)^* = \frac{H_z(i_x, i_y) - H_z(i_x, i_y-1)}{\Delta y}$$

$$C_y^H(i_x, i_y)^* = -\frac{H_z(i_x, i_y) - H_z(i_x-1, i_y)}{\Delta x}$$

Now we aim to find coefficients for the curl equations as multiplication would be less computationally intensive than division. The first curl equation:

$$C_z^E(i_x, i_y) = \left[\frac{1}{\Delta x}\right]\tilde{E}_y(i_x+1, i_y) - \left[\frac{1}{\Delta x}\right]\tilde{E}_y(i_x, i_y) - \left[\frac{1}{\Delta y}\right]\tilde{E}_x(i_x, i_y+1) + \left[\frac{1}{\Delta y}\right]\tilde{E}_x(i_x, i_y)$$
$$= c_{z1}\tilde{E}_y(i_x+1, i_y) - c_{z1}\tilde{E}_y(i_x, i_y) - c_{z2}\tilde{E}_y(i_x, i_y+1) + c_{z2}\tilde{E}_x(i_x, i_y)$$

Where the curl coefficients are:

$$c_{z1} = \frac{1}{\Delta x}, \qquad c_{z2} = \frac{1}{\Delta y}$$

And for the second curl equation,

$$C_x^H(i_x, i_y) = \left[\frac{1}{\Delta y}\right]H_z(i_x, i_y) - \left[\frac{1}{\Delta y}\right]H_z(i_x, i_y-1)$$
$$= c_{z2}H_z(i_x, i_y) - c_{z2}H_z(i_x, i_y-1)$$

And lastly, for the third curl equation,

$$C_y^H(i_x, i_y) = -\left[\frac{1}{\Delta x}\right]H_z(i_x, i_y) + \left[\frac{1}{\Delta x}\right]H_z(i_x-1, i_y)$$
$$= -c_{z1}H_z(i_x, i_y) + c_{z1}H_z(i_x-1, i_y)$$

The standard for the spatial resolution is,

$$\Delta x = \Delta y := \frac{\lambda_{min}}{10}, \qquad \lambda_{min} = \frac{c_{min}}{f}$$

Where c_{min} is classified by the smallest propagation speed of the material, i.e. in the dielectric. The value $\frac{\lambda_{min}}{10}$ is a value determined by how detailed we want our model to look. Changes in the divisor should not influence the propagation behavior of the system, as the time step would change accordingly. And the standard for temporal resolution in 2D is,

$$\Delta t = \frac{1}{v\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2}}$$

Where this corresponds to how quickly light will propagate diagonally across the grid.

All the update coefficients needed for the simulation:

$$c_{z1} = \frac{1}{\Delta x}, \qquad c_{z2} = \frac{1}{\Delta y}$$

$$h_{z1} = \frac{2c_0\epsilon_0\Delta t}{2\mu_{ry}\epsilon_0 + \mu_{ry}\sigma'_y\Delta t}, \quad h_{z2} = \frac{2\mu_{ry}\epsilon_0 - \mu_{ry}\sigma'_y\Delta t}{2\mu_{ry}\epsilon_0 + \mu_{ry}\sigma'_y\Delta t}$$

$$d_{x1} = c_0\Delta t, \quad d_{x2} = 2\sigma_x\eta_0 c_0\Delta t, \quad d_{x3} = \frac{2\sigma'_y\Delta t}{\epsilon_0}$$

$$d_{y1} = 2c_0\Delta t\left(\frac{1}{2} + \frac{\sigma'_y\Delta t}{4\epsilon_0}\right), \qquad d_{y2} = 2\sigma'_y\eta_0(c_0\Delta t)^2$$

$$d_{y3} = 2\sigma_y\eta_0 c_0\Delta t, \qquad d_{y4} = 2\sigma_y\sigma'_y(\eta_0 c_0\Delta t)^2$$

$$e_x = \frac{1}{\epsilon_{rx}}, \qquad e_y = \frac{1}{\epsilon_{ry}}$$

All the iterative steps needed for the simulation shown in order:

$$\mathcal{C}_z^E = c_{z1}\tilde{E}_y(i_x+1, i_y)\big|_t - c_{z1}\tilde{E}_y(i_x, i_y) - c_{z2}\tilde{E}_x(i_x, i_y+1) + c_{z2}\tilde{E}_x(i_x, i_y) \tag{1}$$

$$H_z(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} = -h_{zA}^{i_x,i_y}\mathcal{C}_z^E(i_x, i_y)\big|_t + h_{zB}^{i_x,i_y}H_z(i_x, i_y)\big|_{t-\frac{\Delta t}{2}} \tag{2}$$

$$\mathcal{C}_x^H(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} = c_{z2}H_z(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} - c_{z2}H_z(i_x, i_y-1)\big|_{t+\frac{\Delta t}{2}} \tag{3}$$

$$\widetilde{D}_x(i_x, i_y)\big|_{t+\Delta t} = d_{x1}\mathcal{C}_x^H(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} + d_{x1}\mathcal{C}_x^H(i_x, i_y)\big|_{t-\frac{\Delta t}{2}} - d_{x2}^{i_x,i_y}\tilde{E}_x(i_x, i_y)\big|_t$$
$$+\widetilde{D}_x(i_x, i_y)\big|_{t-\Delta t} - d_{x3}^{i_x,i_y}\widetilde{D}_x(i_x, i_y)\big|_t \tag{4}$$

$$\mathcal{C}_y^H(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} = -c_{z1}H_z(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} + c_{z1}H_z(i_x-1, i_y)\big|_{t+\frac{\Delta t}{2}} \tag{5}$$

$$\widetilde{D}_y(i_x, i_y)\big|_{t+\Delta t} = d_{y1}^{i_x,i_y}\mathcal{C}_y^H(i_x, i_y)\big|_{t+\frac{\Delta t}{2}} + d_{y1}^{i_x,i_y}\mathcal{C}_y^H(i_x, i_y)\big|_{t-\frac{\Delta t}{2}}$$
$$+d_{y2}^{i_x,i_y}\sum_{t_i=\frac{\Delta t}{2}}^{t-\frac{\Delta t}{2}}\mathcal{C}_y^H(i_x, i_y)\big|_{t_i} - d_{y3}^{i_x,i_y}\tilde{E}_y(i_x, i_y)\big|_t$$
$$-d_{y4}^{i_x,i_y}\sum_{t_i=0}^{t}\tilde{E}_y(i_x, i_y)\big|_{t_i} + \widetilde{D}_y(i_x, i_y)\big|_{t-\Delta t} \tag{6}$$

$$\tilde{E}_x(i_x, i_y)\big|_{t+\Delta t} = e_x^{i_x,i_y}\widetilde{D}_x(i_x, i_y)\big|_{t+\Delta t} \tag{7}$$

$$E_y(i_x, i_y)\Big|_{t+\Delta t} = e_y^{i_x, i_y} \widetilde{D}_y(i_x, i_y)\Big|_{t+\Delta t} \tag{8}$$