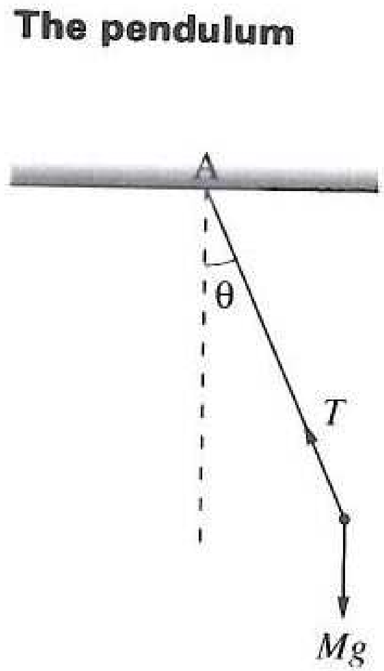
Duffing's equation
Lighthill's technique

Lesson 9

Duffing's equation



$ML\frac{d^2\theta}{dT^2} = -Mg\sin\theta$ with small initial θ_0 at $T = 0$

$u = \frac{\theta}{\theta_0}, \quad \Omega = \sqrt{\frac{g}{L}}$ is a frequency related parameter.

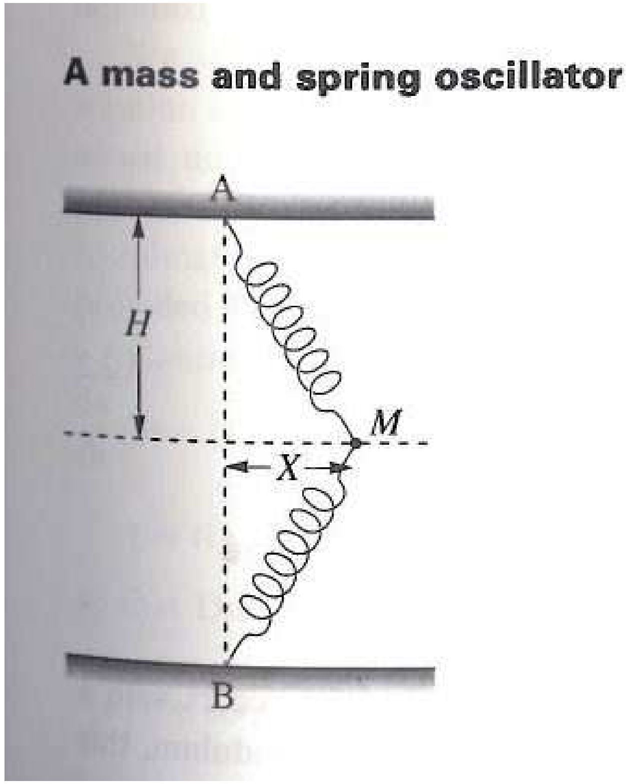
$\frac{d^2u}{dT^2} = -\Omega^2 \frac{\sin(\theta_0 u)}{\theta_0}$ with initial conditions $u = 1$ and $\frac{du}{dT}(0) = 0$. Since θ_0 is small, we change $\sin \theta_0 u$ by $\theta_0 u + \frac{\theta_0^2 u^2}{6}$ and get

$$\frac{d^2u}{dT^2} = -\Omega^2 \left(u - \frac{\theta_0^2 u^2}{6} \right)$$

Introducing the non-dimensional variables $t = \Omega T$ and $\varepsilon = \frac{\theta_0^2}{6}$ we obtain the Duffing's equation

$$\frac{d^2u}{dt^2} + u - \varepsilon u^3 = 0.$$

A mass and spring oscillator



Λ is the spring constant. The spring forces F vary linearly with the extension $F = \Lambda\left(\frac{\sqrt{H^2+X^2}-L}{L}\right)$

A mass and spring oscillator

We have $L < H$. The second Newton's law

$$M\frac{d^2X}{dT^2} = -2F\cos\theta = -\frac{2\Lambda X}{L}\Big(1 - \frac{L}{\sqrt{H^2+X^2}}\Big)$$

with $\cos\theta = \frac{X}{\sqrt{H^2+X^2}}$. Since $X \ll H$ we use

$$\frac{L}{\sqrt{H^2+X^2}} = \frac{L}{H}\Big(1 - \frac{1}{2}\frac{X^2}{H^2}\Big).$$

Introduce nondimensional displacement $u = X/X_0$

$$\frac{d^2u}{dT^2} = u\Big(\frac{2\Lambda(L-H)}{MLH} - \frac{\Lambda X_0^2}{MH^3}u^2\Big) = -\Omega^2\Big(u + \frac{2\Lambda X_0^2}{\Omega^2 MH^3}u^3\Big)$$

A mass and spring oscillator

$$\frac{d^2u}{dT^2} = u\Big(\frac{2\Lambda(L-H)}{MLH} - \frac{2\Lambda X_0^2}{MH^3}u^2\Big) = -\Omega^2\Big(u + \frac{2\Lambda X_0^2}{\Omega^2 MH^3}u^3\Big)$$

where $-\Omega^2 = 2\Lambda(H-L)/MLH$ is the frequency related parameter. We introduce $t = \Omega T$ and $\varepsilon = \frac{2\Lambda X_0^2}{\Omega^2 MH^3} = \frac{LX_0^2}{2H^2(H-L)}$ and get the Duffing's equation

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0.$$

Bounded solutions

We obtained two type of Duffing's equation with $\mp\varepsilon$. We can write

$$\frac{d^2u}{dt^2} + u \mp \varepsilon u^3 = \frac{d}{dt}\Big(\frac{1}{2}\big(\frac{du}{dt}\big)^2 + \frac{1}{2}u^2 \mp \frac{\varepsilon}{4}u^4\Big) = 0.$$

The value

$$\varphi = \frac{1}{2}\big(\frac{du}{dt}\big)^2 + \frac{1}{2}u^2 \mp \frac{\varepsilon}{4}u^4$$

is constant and called the energy with $\frac{1}{2}\big(\frac{du}{dt}\big)^2$ kinetic energy and $\frac{1}{2}u^2 \mp \frac{\varepsilon}{4}u^4$ potential energy. We are looking for bounded solution u.

Bounded solutions

If $\varepsilon > 0$, then $\varphi = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}u^2 + \frac{\varepsilon}{4}u^4 > 0$ and $u \leq u_{\max}$, where

$$\varphi = \frac{1}{2}u_{\max}^2 + \frac{\varepsilon}{4}u_{\max}^4$$

and

$$u_{\max} = \left(-\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 + 4\varphi\varepsilon}\right)^{1/2}.$$

If $\varepsilon < 0$ then it is possible to prove that there is bounded solution for $\varphi \leq \frac{1}{4|\varepsilon|}$.

Lindstedt-Poincaré technique applied to Duffing's equation

Consider the equation $\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$ with $|\varepsilon| \ll 1$, $u(0) = 1$, $\frac{du}{dt}(0) = 0$. Introduce the strained coordinates

$$\tau = t(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots),$$

write the new form of differensial equation

$$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)^2 \frac{d^2u}{d\tau^2} + u + \varepsilon u^3 = 0$$

and use the standard expansion

$$u(\tau, \varepsilon) \sim u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \ldots.$$

Substituting into equation, we have

$$\varepsilon^0: \ \frac{d^2u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \ \frac{du_0}{d\tau}(0) = 0$$

$$\varepsilon^1: \ \frac{d^2u_1}{d\tau^2} + u_1 = -u_0^3 - 2\omega_1 \frac{d^2u_0}{d\tau^2},$$

$$u_1(0) = 0, \ \frac{du_1}{d\tau}(0) = 0,$$

$$\varepsilon^2: \ \frac{d^2u_2}{d\tau^2} + u_2 = -3u_0u_1 - 2\omega_1 \frac{d^2u_1}{d\tau^2} - (\omega_1^2 + 2\omega_2)\frac{d^2u_0}{d\tau^2},$$

$$u_2(0) = 0, \ \frac{du_2}{d\tau}(0) = 0$$

$$u_0 = \cos t.$$

Lindstedt-Poincaré technique

$$\frac{d^2u_1}{d\tau^2}+u_1=-\cos^3\tau+2\omega_1\cos\tau=-\frac{1}{4}\cos 3\tau+(2\omega_1-\frac{3}{4})\cos\tau).$$

We chose $\omega_1=3/8$ to avoid the appearance of secular term since the homogeneous solution is the form $A\cos\tau+B\sin\tau$. Then

$$\frac{d^2u_1}{d\tau^2}+u_1=A\cos\tau+B\sin\tau+\frac{1}{32}\cos 3\tau$$

$$=-\frac{1}{32}\cos\tau+\frac{1}{32}\cos 3\tau.$$

Lindstedt-Poincaré technique

$$\frac{d^2u_2}{d\tau^2} + u_2 = \frac{3}{128}\cos 5\tau + \frac{27}{256}\cos 3\tau + (\frac{21}{128} + 2\omega_2)\cos\tau.$$

We chose $\omega_2 = -21/256$ to avoid the appearance of secular term and get

$$\tau = t(1 + \frac{3}{8}\varepsilon - \frac{21}{256}\varepsilon^2 + O\left(\varepsilon^3)\right) \quad \text{as} \quad \varepsilon \to 0$$

$$u = \cos\tau + \frac{\varepsilon}{32}(\cos 3\tau - \cos\tau) + O(\varepsilon^2).$$

The remainder term $O(\varepsilon^2)$ does not contain the secular term, since we chose ω_2 in a proper form.

Lighthill's technique

The Lighthiill technique is a generalization of Lindstedt-Poincaré in the following sense

$$\tau = t + \varepsilon\omega_1 t + \varepsilon^2\omega_2 t + \ldots \quad \text{change to}$$

$$\tau = t + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots .$$

So the idea is to change the linear functions to more general functions

$$\omega_k t \quad \rightarrow \quad f_k(t).$$

The technically it looks like the following.

Renormalization

Let x be an initial independent variable and $\bar{u}(x,\varepsilon)$ is an original function. We start from the standard expansion

$\bar{u}(x,\varepsilon) \sim \bar{u}_0(x) + \varepsilon \bar{u}_1(x) + \ldots$ that is generally to be nonunif

and then we use the straining transformation

$$x \sim s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \ldots.$$

Substituting the variable x in $\bar{u}(x,\varepsilon)$, we get new expansion

$$u(s,\varepsilon) \sim u_0(s) + \varepsilon u_1(s) + \ldots \quad \text{with new variable } s.$$

Lighthill's condition

Subsequent coefficient functions should be no more singular then previous functions

in both expansions

$$x \sim s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \dots$$

and

$$u(s, \varepsilon) \sim u_0(s) + \varepsilon u_1(s) + \varepsilon^2 u_2(s) + \dots$$

Model problem

We consider the model problem that is related with a shift in the singularity of a differential equation $(x+\varepsilon y)\frac{dy}{dx}+y=0$ with $0<\varepsilon\ll 1$, $y(1)=1$. We start from the standard expansion

$$y(x,\varepsilon)\sim y_0(x)+\varepsilon y_1(x)+\varepsilon^2 y_2(x)+\dots$$

and get

$$y(x,\varepsilon)\sim\frac{1}{x}+\varepsilon\Big(\frac{1}{2x}-\frac{1}{2x^3}\Big)+\varepsilon^2\Big(-\frac{1}{2x^3}+\frac{1}{2x^5}\Big)+\dots.$$

The expansion
$y(x,\varepsilon) \sim \frac{1}{x} + \varepsilon\Big(\frac{1}{2x} - \frac{1}{2x^3}\Big) + \varepsilon^2\Big(-\frac{1}{2x^3} + \frac{1}{2x^5}\Big) + \ldots$ is
nonuniform since

$$y_0 = O\Big(\frac{1}{x}\Big), \quad y_1 = O\Big(\frac{1}{x^3}\Big), \quad y_2 = O\Big(\frac{1}{x^5}\Big), \ \ x \to 0.$$

The region of nonuniformity is $x = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$.
The unperturbed equation is singular when $x = 0$, but the perturbed equation is singular when $x + \varepsilon y = 0$,so the singularity was shifted.

Model problem

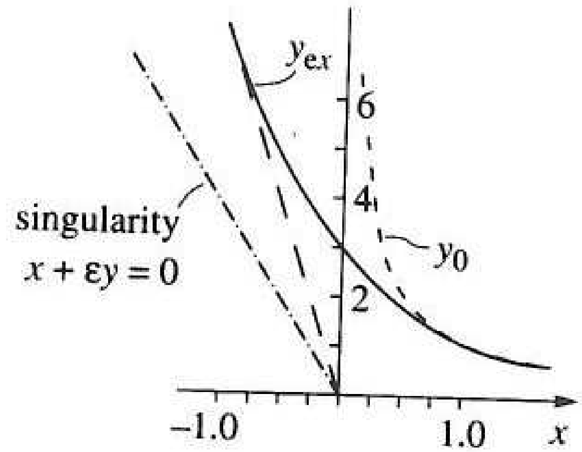


Fig. 3.1 The functions y_{ex} —— and $y_0(x)$ - - - for the case $\varepsilon = 1/4$

$y_{ex} = \frac{-x+\sqrt{x^2+\varepsilon(2+\varepsilon)}}{\varepsilon}$ and $y(0) = \sqrt{1+2/\varepsilon}$. For large x we have $y_{ex} \sim -\frac{2x}{\varepsilon} - \frac{2+\varepsilon}{2x} \rightarrow -\frac{2x}{\varepsilon}$ as $x \rightarrow -\infty$. The behavior of y_0 and y_{ex} is completely different due to the shift of the singularity from $x = 0$ to $x + \varepsilon y = 0$.

Let us substitute the strained coordinates $x \sim s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \dots$ into expansion $y(x, \varepsilon) \sim \frac{1}{x} + \varepsilon\Big(\frac{1}{2x} - \frac{1}{2x^3}\Big) + \varepsilon^2\Big(- \frac{1}{2x^3} + \frac{1}{2x^5}\Big) + \dots$. We collect the coefficient function up to the ε^2. We use

$$\frac{1}{s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \dots} = \frac{1}{s}(1 + \frac{\varepsilon f_1}{s} + \frac{\varepsilon^2 f_2}{s} + \dots)^{-n}$$

$$= \frac{1}{s}\Big(1 - n(\frac{\varepsilon f_1}{s} + \frac{\varepsilon^2 f_2}{s} + \dots) + \frac{-n(-n-1)}{2}(\frac{\varepsilon f_1}{s} + \frac{\varepsilon^2 f_2}{s} + \dots)^2 + .$$

Calculating all coefficient we obtain

$y \sim y_0(s) + \varepsilon y_1(s) + \varepsilon^2 y_2(s) + \ldots$ with

$$y_0(s) = \frac{1}{s}, \qquad y_1(s) = \frac{1}{2s} - \frac{f_1}{s^2} - \frac{1}{2s^3},$$

$$y_2(s) = -\frac{f_2}{s^2} + \frac{f_1^2}{s^3} - \frac{f_1}{2s^2} + \frac{3f_1}{2s^4} - \frac{1}{2s^3} + \frac{1}{2s^5}$$

We need nonincreasing order of singularity, namely $O(\frac{1}{s})$. So $f_1 = -\frac{1}{2s}$.

$$y_2(s) = -\frac{f_2}{s^2} + \frac{1}{4s^5} + \frac{1}{4s^3} - \frac{3}{4s^5} - \frac{1}{2s^3} + \frac{1}{2s^5} \Rightarrow$$

$f_2 = -\frac{1}{4s}$ and $y_2 = 0$.

$$y \sim \frac{1}{s} + \varepsilon\frac{1}{2s} + \varepsilon^2 \cdot 0 + \ldots = \frac{1}{s} + \varepsilon\frac{1}{2s} + O(\varepsilon^2),$$

where

$$x \sim s - \varepsilon\frac{1}{2s} - \varepsilon^2\frac{1}{4s} + O(\varepsilon^3). \tag{1}$$

If $x = O(1)$ (x is away of region of nonuniformity) then $s = O(1)$ and solving (1) we have $s = x + O(\varepsilon)$ and

$$y = \frac{1}{x} + O(\varepsilon),$$

which means that standard expansion and strained coordinates give the same approximation.

$y \sim \frac{1}{s} + \varepsilon \frac{1}{2s} + O(\varepsilon^2)$ where $x \sim s - \varepsilon \frac{1}{2s} - \varepsilon^2 \frac{1}{4s} + O(\varepsilon^3)$. If $x = O(\sqrt{\varepsilon})$ (x is in the region of nonuniformity) then $s = O(\sqrt{\varepsilon})$ and to solve (1) we need to solve

$$x = s - \varepsilon/2s, \quad \Rightarrow \quad s^2 - sx - \varepsilon/2 = 0 \quad \Rightarrow$$

$$s = \frac{x + \sqrt{x^2 + 2\varepsilon}}{2}$$

where we take the root associated with $s = x + O(\varepsilon)$. Then

$$y \sim \frac{1}{s} = \frac{2}{x + \sqrt{x^2 + 2\varepsilon}} \to \sqrt{\frac{2}{\varepsilon}} \text{ as } \; x = 0.$$

Model problem

Let us compare $y \sim \frac{1}{s} = \frac{2}{x+\sqrt{x^2+2\varepsilon}} \to \sqrt{\frac{2}{\varepsilon}}$ as $x = 0$ with exact solution

$$y_{ex} = \frac{-x + \sqrt{x^2 + \varepsilon(2+\varepsilon)}}{\varepsilon}|_{x=0} = \sqrt{1 + \frac{2}{\varepsilon}}$$

$$= \sqrt{\frac{2}{\varepsilon}}(1 + \frac{\varepsilon}{2})^{1/2} = \sqrt{\frac{2}{\varepsilon}}(1 + O(\varepsilon)).$$

Model problem

The choice of the coefficients in $x \sim s - \varepsilon f_1 + O(\varepsilon^2)$ is not unique. Let us choose $f_1 = -\frac{1}{2s} + \frac{s}{2}$. Then $y_1 = 0$ (but it does not important) and

$$x = s + \varepsilon\Big(- \frac{1}{2s} + \frac{s}{2}\Big) + O(\varepsilon^2).$$

If $x = O(\sqrt{\varepsilon})$ (x is in the region of nonuniformity) then $s = O(\sqrt{\varepsilon})$ and to express s we need to solve

$$s^2\Big(1 + \frac{\varepsilon}{2}\Big) - sx - \frac{\varepsilon}{2} = 0 \quad \Rightarrow \quad s = \frac{x + \sqrt{x^2 + \varepsilon(2+\varepsilon)}}{2+\varepsilon}.$$

Since $\sqrt{x^2+\varepsilon(2+\varepsilon)} \sim \sqrt{x^2+2\varepsilon}$ and $\frac{1}{2+\varepsilon} \sim \frac{1}{2}$ we have

$$s = \frac{x+\sqrt{x^2+2\varepsilon}}{2} + O(\varepsilon)$$

and

$$y \sim \frac{1}{s} = \frac{2}{x+\sqrt{x^2+2\varepsilon}} \to \sqrt{\frac{2}{\varepsilon}} \text{ as } x = 0$$

as in the previous case.

The end