Strained Coordinates

Lesson 8

Strained coordinates

The method of strained coordinates is a technique for dealing with certain types of nonuniformities which occur in asymptotic expansions.

The Lindstedt-Poincaré technique

The technique that we use in the previous two chapters is a so called straightforward asymptotic expansion.

$$f(x,\varepsilon) = \sim \sum_{n=0}^{\infty} f_n(t)\delta_n(\varepsilon) \quad \text{as} \quad \varepsilon \to 0,$$

where the coefficients can be found by

$$f_N(t) = \lim_{\varepsilon \to 0} \left(\frac{f(t,\varepsilon) - \sum_{n=0}^{N-1} f_n(t)\delta_n(\varepsilon)}{\delta_N(\varepsilon)} \right)$$

The limit is taken when $\varepsilon \to 0$ and t is fixed.

The Lindstedt-Poincaré technique

Let us compare two expansions.

$$\cos(t+\varepsilon) \sim \cos t - \varepsilon \sin t - \frac{\varepsilon^2}{2}\cos t + \ldots$$

and

$$\cos(t+\varepsilon(1+t)) \sim \cos t - \varepsilon(1+t)\sin t - \frac{\varepsilon^2(1+t)^2}{2}\cos t + \ldots$$

The first expansion is uniform and the second expansion is nonuniform. The region of the nonuniformity occurs when $\varepsilon t = O(1)$ or $t = O(1/\varepsilon)$ as $\varepsilon \to 0$

The Lindstedt-Poincaré technique

Let us make the following change of variables $\tau = t(1+\varepsilon)$. Then

$$\cos(t + \varepsilon(1+t)) = \cos(t(1+\varepsilon) + \varepsilon) = \cos(\tau + \varepsilon) = F(\tau, \varepsilon)$$

$$\cos(\tau + \varepsilon) = F(\tau, \varepsilon) \sim \cos\tau - \varepsilon \sin\tau - \frac{\varepsilon^2}{2}\cos\tau + \dots .$$

We get expansions of the form

$$F(\tau, \varepsilon) \sim \sum_{n=0}^{\infty} F_n(\tau)\delta_n(\varepsilon),$$

The Lindstedt-Poincaré technique

$F(\tau,\varepsilon) \sim \cos\tau - \varepsilon\sin\tau - \frac{\varepsilon^2}{2}\cos\tau + \ldots.$

$F(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} F_n(\tau)\delta_n(\varepsilon),$

$$F_N(\tau) = \lim_{\varepsilon\to 0}\left(\frac{F(\tau,\varepsilon) - \sum_{n=0}^{N-1} F_n(\tau)\delta_n(\varepsilon)}{\delta_N(\varepsilon)}\right)$$

The limit is taken when $\varepsilon \to 0$ and τ is fixed. The new variable $\tau = t(1+\varepsilon)$ is called *strained coordinate*

relative error $\frac{\tau}{t} = 1 + \varepsilon$ is small

numerical error $\tau - t = \varepsilon t$ is large if $t = O(1/\varepsilon)$

Model problem

Consider the differential equation $\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon u$ for $t > 0$ with initial conditions $u(0) = a$, $\frac{du}{dt}(0) = 0$. We first construct the straightforward expansion

$$u(t, \varepsilon) = u_0(t) + \varepsilon u_1(t) + \dots$$

Substituting to the equation, we obtain

$$\frac{d^2u_0}{dt^2} + \varepsilon \frac{d^2u_1}{dt^2} + \dots + \omega_0^2(u_0 + \varepsilon u_1 + \dots) \sim \varepsilon u_0 + \dots$$

The initial conditions are

$$u_0(0) = a, \quad u_1(0) = 0, \qquad \frac{du_0}{dt}(0) = 0, \quad \frac{du_1}{dt}(0) = 0 \quad \dots.$$

Model problem

$\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon u$

$$\varepsilon^0 : \quad \frac{d^2u_0}{dt^2} + \omega_0^2 u_0 = 0, \quad u_0(0) = a, \ \frac{du_0}{dt}(0) = 0$$

$$\varepsilon^1 : \quad \frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = u_0, \quad u_1(0) = 0, \ \frac{du_1}{dt}(0) = 0,$$

$$u_0 = a \cos \omega_0 t$$

is the solution of the first equation and unperturbed problem.

Model problem

$\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = a\cos\omega_0 t$

$A\cos\omega_0 t + B\sin\omega_0 t$ is the homogeneous solution

$\alpha t\sin\omega_0 t + \beta t\cos\omega_0 t,\ (\beta = 0)$ is a particular solution

$2\alpha\omega_0\cos\omega_0 t = a\cos\omega_0 t \quad \Rightarrow \quad \alpha = \frac{a}{2\omega_0}$

$$u_1 = A\cos\omega_0 t + B\sin\omega_0 t + \frac{at}{2\omega_0}\sin\omega_0 t \text{ with } A = B = 0.$$

$$u \sim a\cos\omega_0 t + \varepsilon\frac{at}{2\omega_0}\sin\omega_0 t \text{ with secular term}$$

when $t = O(1/\varepsilon)$.

Model problem

Let us compare with the exact solution of $\frac{d^2u}{dt^2} + (\omega_0^2 - \varepsilon)u = 0$:

$$u = a\cos(\sqrt{\omega_0^2 - \varepsilon t}) = a\cos(\omega_0\sqrt{1 - \varepsilon/\omega_0^2 t}).$$

The solution is bounded with the constant amplitude a. Why the expansion give the secular term? To see this we analyze the series of exact solution in terms of ε

$$\sqrt{1 - \varepsilon/\omega_0^2} = 1 - \frac{1}{2}\frac{\varepsilon}{\omega_0^2} + \ldots$$

Then

Model problem

$$
\begin{aligned}
&a\cos\left(\omega_0(1-\frac{\varepsilon}{2\omega_0^2}+\ldots)t\right)\\
&=a\cos\omega_0 t\cos\left(\frac{\varepsilon t}{2\omega_0}+\ldots\right)+a\sin\omega_0 t\sin\left(\frac{\varepsilon t}{2\omega_0}+\ldots\right)\\
&=a\cos\omega_0 t\left(1-\frac{\varepsilon^2 t^2}{8\omega_0^2}+\ldots\right)+a\sin\omega_0 t\left(\frac{\varepsilon t}{2\omega_0}+\ldots\right)\\
&=a\cos\omega_0 t+\varepsilon\frac{at}{2\omega_0}\sin\omega_0 t+\ldots
\end{aligned}
$$

Is the last expansion is asymptotic expansion?

Model problem

The exact solution shows that the effect of perturbation is to modify ω_0 to $\omega_0\sqrt{1-\varepsilon/\omega_0^2}$. The strained coordinate $\tau = t\sqrt{1-\varepsilon/\omega_0^2}$ provides the uniform expansion

$$u = a\cos\omega_0\tau$$

with vanishing remainder term. How to know the strained coordinate without knowledge of exact solution. The Lindstedt-Poincaré technique gives the answer.

Model problem

Set $\tau = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)t$. Then

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)\frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)^2\frac{d^2}{d\tau^2}$$

$\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon u \quad \Rightarrow$

$(1 + 2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_2) + \ldots)\frac{d^2u(\tau)}{d\tau^2} + \omega_0^2 u(\tau) = \varepsilon u(\tau)$.

Let us construct the expansion for $u(\tau, \varepsilon)$

Model problem

$$(1+2\varepsilon\omega_1+\varepsilon^2(\omega_1^2+2\omega_2)+\ldots)\frac{d^2u(\tau)}{d\tau^2}+\omega_0^2u(\tau)=\varepsilon u(\tau)$$

$$u(\tau,\varepsilon)\sim u_0(\tau)+\varepsilon u_1(\tau)+\varepsilon^2u_2(\tau)+\ldots$$

$$u(\tau)|_{\tau=0}=\bar{u}(t)|_{t=0}=0,$$

$$(1+\varepsilon\omega_1+\varepsilon^2\omega_2+\ldots)\frac{du(\tau)}{d\tau}|_{\tau=0}=\frac{du(t)}{dt}|_{t=0}=0$$

Model problem

$$\varepsilon^0 : \; \frac{d^2u_0}{d\tau^2} + \omega_0^2 u_0 = 0, \quad u_0(0) = a, \; \frac{du_0}{d\tau} = 0$$

$$\varepsilon^1 : \; \frac{d^2u_1}{d\tau^2} + \omega_0^2 u_1 + 2\omega_1 \frac{d^2u_0}{d\tau^2} = u_0,$$

$$u_1(0) = 0, \; \frac{du_1}{d\tau}(0) + \omega_1 \frac{du_0}{d\tau} = 0,$$

$$\varepsilon^2 : \; \frac{d^2u_2}{d\tau^2} + \omega_0^2 u_2 + (\omega_1^2 + 2\omega_2) \frac{d^2u_0}{d\tau^2} + 2\omega_1 \frac{d^2u_1}{d\tau^2} = u_1,$$

$$u_2(0) = 0, \; \frac{du_2}{d\tau}(0) + \omega_1 \frac{du_1}{d\tau} + \omega_2 \frac{du_0}{d\tau} = 0$$

Model problem

$$\frac{d^2u_0}{d\tau^2} + \omega_0^2 u_0 = 0 \quad \Rightarrow \quad u_0 = a\cos\omega_0\tau,$$

$$\frac{d^2u_1}{d\tau^2} + \omega_0^2 u_1 = -2\omega_1 \frac{d^2u_0}{d\tau^2} + u_0 = (1 + 2\omega_1\omega_0^2)a\cos\omega_0\tau$$

$$\Rightarrow \quad \omega_1 = -1/2\omega_0^2 \text{ and } u_1(\tau) = 0$$

$$\frac{d^2u_2}{d\tau^2} + \omega_0^2 u_2 = (\frac{1}{4\omega_0^4} + 2\omega_2)a\omega_0^2\cos\omega_0\tau$$

$$\Rightarrow \quad \omega_2 = -1/8\omega_0^4 \text{ and } u_2(\tau) = 0$$

The strained coordinate is given by

$$\tau = t\Big(1 - \frac{\varepsilon}{2\omega_0^2} - \frac{\varepsilon^3}{8\omega_0^4} + \dots\Big).$$

The expansion is $u(\tau, \varepsilon) = a\cos\omega_0\tau$.

The exact solution is $a\cos\left(\omega_0(1 - \frac{\varepsilon}{2\omega_0^2} - \frac{\varepsilon^2}{8\omega_0^4} + \dots)\right)$

The Lindstedt-Poincaré technique generates the expansion by determining the constants ω_n from the requirement that secular terms are absent from the expansion.

The end