Uniform and nonuniform expansions

Lesson 7

Uniform and nonuniform expansions

If

$$f(x,\varepsilon) = \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) + R_N(x,\varepsilon)$$

is an asymptotic expansion then

$$R_N(x,\varepsilon) = O(\delta_{N+1}(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0.$$

Or we can say

$$|R_N(x,\varepsilon)| \leq K|\delta_{N+1}(\varepsilon)|.$$

Example of uniform expansion

$\frac{1}{1-\varepsilon\sin x} \sim 1 + \varepsilon \sin x + \varepsilon^2 \sin^2 x + \varepsilon^3 \sin^3 x + \ldots$ with

$$R_N = \sum_{n=N+1}^{\infty} \varepsilon^n \sin^n x \quad \text{so} \quad \lim_{\varepsilon \to 0} \frac{R_N}{\varepsilon^{N+1}} = \sin^{N+1} x$$

and we can estimate

$$|R_N(x,\varepsilon)| \le K|\varepsilon^{N+1}|$$

with any $K > 1$.

Example of nonuniform expansion

$\frac{1}{1-\varepsilon \sin x} \sim 1 + \varepsilon x + \varepsilon^2 x^2 + \varepsilon^3 x^3 + \ldots$ with

$$R_N = \sum_{n=N+1}^{\infty} \varepsilon^n x^n \quad \text{so} \quad \lim_{\varepsilon \to 0} \frac{R_N}{\varepsilon^{N+1}} = x^{N+1}$$

Since x is not bounded we can not find K such that

$$|R_N(x,\varepsilon)| \leq K|\varepsilon^{N+1}|.$$

Region of nonuniformity

The principal idea of the asymptotic expansion that the subsequent term in the expansion smaller the previous one $a_{n+1}(x)\delta_{n+1}(\varepsilon) = o\big(a_n(x)\delta_n(\varepsilon)\big)$.

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$\frac{1}{1-\varepsilon\sin x} \sim 1 + \varepsilon x + \varepsilon^2 x^2 + \varepsilon^3 x^3 + \ldots$ When two subsequent term has the same order?

$$\varepsilon^{n+1}x^{n+1} = O(\varepsilon^n x^n) \quad \Rightarrow \quad \varepsilon x = O(1)$$

or $x = O(1/\varepsilon)$. If $\varepsilon^{n+1}x^{n+1} = o(\varepsilon^n x^n) \;\Rightarrow\; \varepsilon x = o(1)$
or, for instance, $x = O(\frac{1}{\sqrt{\varepsilon}})$

Region of nonuniformity

- $1 + \varepsilon e^x + \varepsilon^2 e^{2x} + \varepsilon^3 e^{3x} + \dots$

$$\varepsilon e^x = O(1) \quad \Rightarrow \quad e^x = O(1/\varepsilon)$$

or $x = O(-\ln \varepsilon)$ as $\varepsilon \to 0$ and x is large.

Region of nonuniformity

- $1 + \varepsilon e^x + \varepsilon^2 e^{2x} + \varepsilon^3 e^{3x} + \dots$

$$\varepsilon e^x = O(1) \quad \Rightarrow \quad e^x = O(1/\varepsilon)$$

or $x = O(-\ln \varepsilon)$ as $\varepsilon \to 0$ and x is large.

- $1 + \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \frac{\varepsilon^3}{x^3} + \dots$If

$$\frac{\varepsilon}{x} = O(1) \quad \Rightarrow \quad x = O(\varepsilon) \;\; \text{as} \;\; \varepsilon \to 0$$

for small values of x.

Region of nonuniformity (more examples)

$$\sin(x+\varepsilon) = \sin x \cos\varepsilon + \cos x \sin\varepsilon$$

$$= \sin x\left(1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)\right) + \cos x\left(\varepsilon - \frac{\varepsilon^3}{6} + O(\varepsilon^5)\right)$$

$$= \sin x + \varepsilon\cos x - \frac{\varepsilon^2}{2}\sin x - \frac{\varepsilon^3}{6}\cos x + O(\varepsilon^4)$$

$$\sin(x(1+\varepsilon)) = \sin x \cos\varepsilon x + \cos x \sin\varepsilon x$$

$$= \sin x\left(1 - \frac{\varepsilon^2 x^2}{2} + O(\varepsilon^4 x^4)\right) + \cos x\left(\varepsilon x - \frac{\varepsilon^3 x^3}{6} + O(\varepsilon^5 x^5)\right)$$

$$= \sin x + \varepsilon x\cos x - \frac{\varepsilon^2 x^2}{2}\sin x - \frac{\varepsilon^3 x^3}{6}\cos x + O(\varepsilon^4 x^4)$$

Region of nonuniformity

$$\sin(x+\varepsilon) = \sin x + \varepsilon\cos x - \frac{\varepsilon^2}{2}\sin x - \frac{\varepsilon^3}{6}\cos x + O(\varepsilon^4)$$

is the uniform expansion and

$$\sin(x(1+\varepsilon)) = \sin x + \varepsilon x\cos x - \frac{\varepsilon^2 x^2}{2}\sin x - \frac{\varepsilon^3 x^3}{6}\cos x + O(\varepsilon^4 x^4)$$

is nonuniform because of appearance of x in coefficients.

Region of nonuniformity

- The expansion $f(x,\varepsilon) \sim \sum_{n=0}^{\infty} f_n(x)\delta_n(\varepsilon)$ as $\varepsilon \to 0$ is uniform if coefficients $f_n(x)$ are bounded.

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- Is it true that if $f_n(x)$ are unbounded then the expantion is nonuniform?
- The expansion is uniform $x + \varepsilon x + \varepsilon^2 x + \varepsilon^3 x + \dots$
- For the nonuniformity is required $f_{n+1}(x)\delta_{n+1}(\varepsilon) = O\big(f_n(x)\delta_n(\varepsilon)\big) \quad \Rightarrow$ $f_{n+1}(x) = f_n(x) O\left(\frac{\delta_n(\varepsilon)}{\delta_{n+1}(\varepsilon)}\right)$ The coefficient $f_{n+1}(x)$ increase faster then $f_n(x)$ since $\frac{\delta_n(\varepsilon)}{\delta_{n+1}(\varepsilon)}$ increase.

Sources of nonuniformity

- Infinite domains which allow long term effects of small perturbations to accumulate

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- Infinite domains which allow long term effects of small perturbations to accumulate
- Singularities in governing equations which lead to localized regions of rapid change

Infinite domains

Consider nonlinear oscillator equation $\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$ for $t > 0$ with initial conditions $u(0) = a$, $\frac{du}{dt}(0) = b$. We use the standard expansion

$$u(t, \varepsilon) = u_0(t) + \varepsilon u_1(t) + \ldots$$

Substituting to the equation, we obtain

$$\left(\frac{d^2u_0}{dt^2} + u_0\right) + \varepsilon\left(\frac{d^2u_1}{dt^2} + u_1 + u_0^3\right) + O(\varepsilon^2) = 0$$

$$u_0(0) + \varepsilon u_1(0) + \ldots = a, \qquad \frac{du_0}{dt}(0) + \varepsilon\frac{du_1}{dt}(0) + \ldots = b$$

We need to solve the following equations

$$\varepsilon^0 : \quad \frac{d^2u_0}{dt^2} + u_0 = 0, \quad u_0(0) = a, \ \frac{du_0}{dt}(0) = b \quad \Rightarrow \quad u_0 = a\cos t,$$

$$\varepsilon^1 : \quad \frac{d^2u_1}{dt^2} + u_1 + u_0^3 = 0, \quad u_1(0) = 0, \ \frac{du_1}{dt}(0) = 0 \quad \Rightarrow$$

$$\frac{d^2u_1}{dt^2} + u_1 = -a^3\cos^3 t = -\frac{a^3}{4}\cos 3t - \frac{3a^3}{4}\cos t.$$

Infinite domains

$$\frac{d^2u_1}{dt^2} + u_1 = -\frac{a^3}{4}\cos 3t - \frac{3a^3}{4}\cos t$$

$A\cos t + B\sin t$ is a homogeneous solution,

$\alpha\cos 3t + \beta\sin 3t$ is a particular solution corresponding to $\cos 3t \quad \Rightarrow \quad \alpha = \frac{a^3}{32},\ \beta = 0$,

$\delta t\cos t + \gamma t\sin t$ is a particular solution corresponding to $\cos t$ since $\cos t$ has the same form as a general solution. We have $\delta = 0,\ \gamma = -\frac{3a^3}{8}$.

The general solution is

$u_1 = A\cos t + B\sin t + \frac{a^3}{32}\cos 3t - \frac{3a^3}{8} t\sin t$,

$u_1(0) = \frac{du_1}{dt}(0) = 0$.

$$A = -\frac{a^3}{32},\ B = 0 \quad \Rightarrow \quad u_1 = \frac{a^3}{32}(\cos 3t - \cos t) - \frac{3a^3}{8} t\sin t,$$

$$u \sim a\cos t + \varepsilon\left(\frac{a^3}{32}(\cos 3t - \cos t) - \frac{3a^3}{8} t\sin t\right) + \dots$$

The term $t\sin t$ is called *secular* term. It is an oscillatory term of increasing amplitude. It leads to the nonuniformity. $\cos t = O(\varepsilon t\sin t) \quad \Rightarrow \quad t = O(1/\varepsilon)$ as $\varepsilon \to 0$.

Small parameter multiplying the highest derivative

Consider the equation $\varepsilon\frac{dy}{dx}+y=e^{-x}$ for $x>0$, $\varepsilon\ll 1$ with initial conditions $y(0)=2$. We use the standard expansion

$$y(x,\varepsilon)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2 y_2(x)+\dots$$

Substituting to the equation, we obtain

$$\varepsilon\Big(\frac{dy_0}{dx}+\varepsilon\frac{dy_1}{dx}+\dots\Big)+y_0+\varepsilon y_1+\varepsilon^2 y_2+O(\varepsilon^3)=e^{-x}$$

$$y_0(0)+\varepsilon y_1(0)+\varepsilon^2 y_2(0)+\dots=2$$

Small parameter multiplying the highest derivative

$\varepsilon \frac{dy}{dx} + y = e^{-x}$

$$\varepsilon^0: \ y_0 = e^{-x}, \quad y_0(0) = 2,$$

$$\varepsilon^1: \ y_1 = -\frac{dy_0}{dx} = e^{-x}, \quad y_1(0) = 0,$$

$$\varepsilon^2: \ y_2 = -\frac{dy_1}{dx} = e^{-x}, \quad y_2(0) = 0$$

We obtain the expansion $y \sim e^{-x} + \varepsilon e^{-x} + \varepsilon^2 e^{-x} + \ldots$, but the boundary conditions can not be satisfied. The unperturbed equation ($\varepsilon = 0$) is an algebraic equation. The nature of an differential equation and algebraic equation is very different.

Small parameter multiplying the highest derivative

Let us compare with the exact solution of

$\varepsilon \frac{dy}{dx} + y = e^{-x}$: $y_{ex} = \frac{1-2\varepsilon}{1-\varepsilon} e^{-x/\varepsilon} + \frac{e^{-x}}{1-\varepsilon}$

$y_{ex} = (1 - \varepsilon - \varepsilon^2 + \ldots)e^{-x/\varepsilon} + (1 + \varepsilon + \varepsilon^2 + \ldots)e^{-x} = I + II.$

The expansion generates the second term II, but fails to create I.

$$e^{-0/\varepsilon} = 1$$

and

$$e^{-x/\varepsilon} \text{ decays rapidly for positive } x$$

Small parameter multiplying the highest derivative

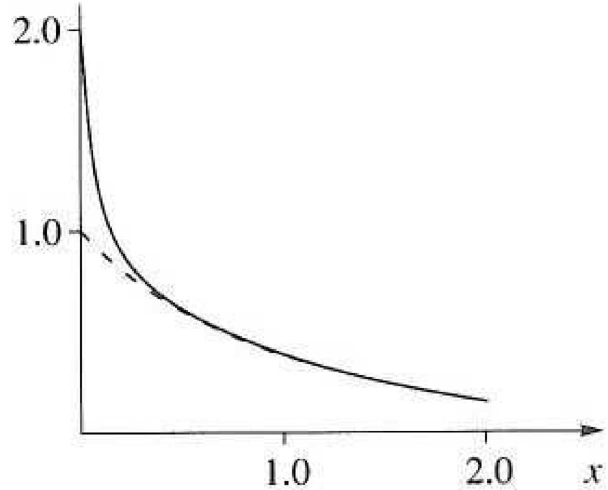


Fig. 2.2 y_{ex} —— and y_0 - - - for $\varepsilon = 0.05$

The first term behaves as $e^{-x/\varepsilon} = o(\varepsilon^n)$ as $\varepsilon \to 0$, $\forall n$, if $x = O(1)$ and $e^{-x/\varepsilon} = O(1)$ as $\varepsilon \to 0$ if $x = O(\varepsilon)$. The region near $x = 0$ is called *boundary layer*.

Other example

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0,\ x \in (0,1),\ \varepsilon \ll 1,\ y(0) = 0,\ y(1) = 1.$$

$y = e^{1-x} - e^{-x/\varepsilon} e^{1+x} + O(\varepsilon)$ is the exact solution.

If $x \nsim 0$ the behavior of the solution define e^{1-x}, because $e^{-x/\varepsilon}$ is negligible. For $x \sim 0$ $e^{-x/\varepsilon} \nearrow 1$ rapidly

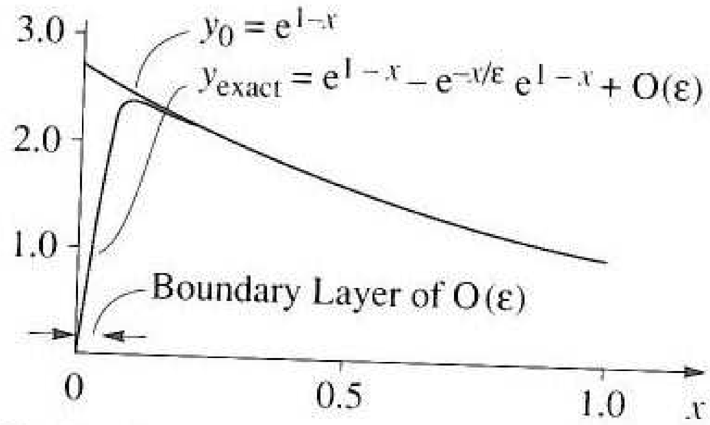


Fig. 2.3 Comparison of y_{exact} and y_0

Other example

$\varepsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0$,

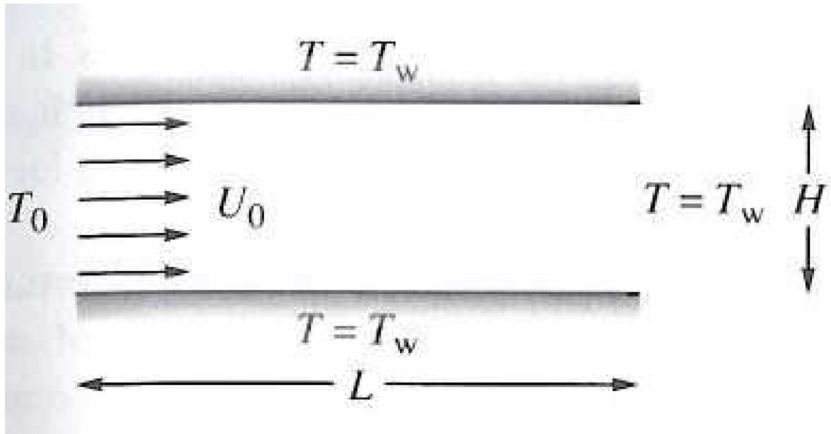
$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \ldots$$

$$\varepsilon^0 : \quad \frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 0, \; y_0(1) = 1$$

$$\varepsilon^1 : \quad \frac{dy_1}{dx} + y_1 = -\frac{d^2 y_0}{dx^2}, \quad y_1(0) = 0, \; y_1(1) = 0,$$

Both of boundary conditions can not be satisfied. If we satisfy $y_0(1) = 1$ then $y_0 = e^{1-x}$ is a good approximation away of boundary layer. If we satisfy $y_0(0) = 0$ we get $y_0 = 0$, that approximate the solution only at $x = 0$.

Once more example



$$U_0 \frac{\partial T}{\partial X} = \alpha(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2})$$

$$x = \frac{X}{L}, y = \frac{Y}{H}, \theta = \frac{T - T_w}{T_0 - T_w} \;\Rightarrow\; \varepsilon \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} - Pe \frac{\partial \theta}{\partial x} = 0$$

with $\varepsilon = \frac{H^2}{L^2}$ and $Pe = \frac{U_0 H^2}{\alpha L}$.

Once more example

We have $\varepsilon\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - Pe\frac{\partial\theta}{\partial x} = 0$ with boundary conditions

$\theta = 0$ on $y = 0, 1,\ \&\ x = 1$ (outlet), $\quad \theta = 1$ on $x = 0$ (inlet)

Using $\theta(x, y, \varepsilon) \sim \theta_0(x, y) + \varepsilon\theta_1(x, y) + \ldots$ we get

$$\frac{\partial^2\theta_0}{\partial y^2} - Pe\frac{\partial\theta_0}{\partial x} = 0$$

All initial conditions can not be satisfied, since the initial equation is of the elliptic type, but the equation for θ_0 is of the parabolic type. The boundary layer appears on $x = 1$.

The end