Asymptotic sequence and expansions

Lesson 6

Definition of asymptotic expansion as $x \to \infty$

The expansion $f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_N}{x^N} + R_N$ is an asymptotic expansion if for any N

$$R_N = O(\frac{1}{x^{N+1}}) \qquad x \to \infty$$

In this case we write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n} \text{ as } x \to \infty \text{ (not } n \to \infty)$$

$$\lim_{N\to\infty} R_N = \infty \ \forall x \text{ fixed}$$

$$\lim_{x\to\infty} R_N = 0 \ \forall N \text{ fixed}$$

Finding the coefficients

- How to find $a_0, a_1, a_2, \ldots$?

  Since $f(x) = a_0 + O\left(\frac{1}{x}\right)$, then $a_0 = \lim_{x\to\infty} f(x)$

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  Since $f(x) = a_0 + O\left(\frac{1}{x}\right)$, then $a_0 = \lim_{x \to \infty} f(x)$
- From $f(x) = a_0 + \frac{a_1}{x} + O\left(\frac{1}{x^2}\right)$ we get $x(f(x) - a_0) = a_1 + O\left(\frac{1}{x}\right)$. So

$$a_1 = \lim_{x \to \infty} x(f(x) - a_0)$$

Finding the coefficients

- Further, from $f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + O\left(\frac{1}{x^3}\right)$

$a_2 = \lim_{x\to\infty} x^2\left(f(x) - a_0 - \frac{a_1}{x}\right)$

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- In general

$$a_N = \lim_{x\to\infty} x^N\left(f(x) - \sum_{n=0}^{N-1} \frac{a_n}{x^n}\right)$$

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- $\{1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots\}$ is an asymptotic sequence

Asymptotic sequences

If we write $\varepsilon = \frac{1}{x}$ for $x \sim \infty$, then

$$\{1, \varepsilon, \varepsilon^1, \varepsilon^2, \varepsilon^3, \ldots\}$$

$$\{1, \varepsilon^{1/2}, \varepsilon^1, \varepsilon^{3/2}, \varepsilon^2, \ldots\}$$

en general if $\delta_n(\varepsilon)$ is a term of asymptotic sequence then

$$\{\delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon), \ldots\}$$

$$\delta_{n+1}(\varepsilon) = o(\delta_n(\varepsilon)) \quad \varepsilon \longrightarrow 0$$

Examples of asymptotic sequences

$$\{1, \sin\varepsilon, \sin^2\varepsilon, \sin^3\varepsilon, \ldots, \sin^n\varepsilon, \ldots\}$$

$$\{1, \ln(1+\varepsilon), \ln(1+\varepsilon^2), \ldots \ln(1+\varepsilon^n), \ldots\}$$

$$\{\tan(\varepsilon^{1/2}), \tan(\varepsilon), \tan(\varepsilon^{3/2}), \ldots, \tan(\varepsilon^{n/2}), \ldots\}$$

$$\{\frac{\ln\varepsilon}{\varepsilon}, \frac{1}{\varepsilon}, \ln\varepsilon, 1, \varepsilon\ln\varepsilon, \varepsilon, \varepsilon^2\ln\varepsilon, \varepsilon^2, \ldots\}$$

Expansion in terms of $\delta_n(\varepsilon)$

$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n\delta_n(\varepsilon)$ that means

$$f(\varepsilon) = \sum_{n=0}^{N} a_n\delta_n(\varepsilon) + R_N, \quad R_N = O(\delta_{n+1}(\varepsilon)) \;\; \varepsilon \to 0$$

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$$a_N = \lim_{\varepsilon \to 0} \left(\frac{f(\varepsilon) - \sum_{n=0}^{N-1} a_n \delta_n(\varepsilon)}{\delta_N(\varepsilon)} \right)$$

Positive example

Let us decompose the function e^ε using $\delta_n(\varepsilon) = \ln(1+\varepsilon^n)$, e. g.

$$e^\varepsilon \sim a_0 + a_1 \ln(1+\varepsilon) + a_2 \ln(1+\varepsilon^2) + \ldots$$

$$a_0 = 1$$

$e^\varepsilon - 1 = a_1 \ln(1+\varepsilon) + o(\ln(1+\varepsilon))$,

$$\lim_{\varepsilon\to 0}\Big(\frac{e^\varepsilon - 1}{\ln(1+\varepsilon)}\Big) = a_1 + \lim_{\varepsilon\to 0}\Big(\frac{o(\ln(1+\varepsilon))}{\ln(1+\varepsilon)}\Big)$$

$$a_1 = \lim_{\varepsilon\to 0}\Big(\frac{e^\varepsilon - 1}{\ln(1+\varepsilon)}\Big) = \lim_{\varepsilon\to 0}\Big(\frac{1+\varepsilon+O(\varepsilon^2)-1}{\varepsilon+O(\varepsilon^2)}\Big) = 1$$

Positive example

$$e^{\varepsilon} - 1 - \ln(1+\varepsilon) = a_2 \ln(1+\varepsilon^2) + o(\ln(1+\varepsilon^2)),$$

$$a_2 = \lim_{\varepsilon\to 0}\left(\frac{e^{\varepsilon} - 1 - \ln(1+\varepsilon)}{\ln(1+\varepsilon^2)}\right)$$

$$= \lim_{\varepsilon\to 0}\left(\frac{1+\varepsilon+\varepsilon^2/2+O(\varepsilon^3) - 1 - \varepsilon + \varepsilon^2/2}{\varepsilon^2 + O(\varepsilon^3)}\right) = 1$$

$$e^{\varepsilon} = 1 + \ln(1+\varepsilon) + \ln(1+\varepsilon^2) - \frac{1}{6}\ln(1+\varepsilon^3) + o(\ln(1+\varepsilon^3))$$

Negative example

Let us attempt to expand $\cos(\varepsilon^{1/2}+\varepsilon)$ using the same asymptotic equation $\delta_n(\varepsilon) = \ln(1+\varepsilon^n)$

$$\cos(\varepsilon^{1/2}+\varepsilon) \sim a_0 + a_1 \ln(1+\varepsilon) + a_2 \ln(1+\varepsilon^2) + \dots$$

$$a_0 = \cos(\varepsilon^{1/2}+\varepsilon) = 1$$

$$a_1 = \lim_{\varepsilon\to 0}\Big(\frac{\cos(\varepsilon^{1/2}+\varepsilon)-1}{\ln(1+\varepsilon)}\Big)$$

$$= \lim_{\varepsilon\to 0}\Big(\frac{1-(\varepsilon^{1/2}+\varepsilon)^2/2+O(\varepsilon^2)-1}{\varepsilon+O(\varepsilon^2)}\Big) = -\frac{1}{2}$$

Negative example

$$a_2 = \lim_{\varepsilon \to 0} \Big(\frac{\cos(\varepsilon^{1/2} + \varepsilon) - 1}{\ln(1+\varepsilon)} \Big)$$

$$= \lim_{\varepsilon \to 0} \Big(\frac{1 - (\varepsilon + 2\varepsilon^{3/2} + \varepsilon^2)/2 + O(\varepsilon^2) - 1 + \varepsilon/2 - \varepsilon^2/4}{\varepsilon^2 + O(\varepsilon^4)} \Big)$$

$$= \lim_{\varepsilon \to 0} \Big(- \frac{\varepsilon^{3/2} + O(\varepsilon^2)}{\varepsilon^2 + O(\varepsilon^4)} \Big) = -\infty$$

What is the solution of the problem? Add the term $\ln(1+\varepsilon^{3/2})$ and its subsequent powers.

Uniqueness of expansions

- Any function $f(x)$ produce unique expansion.

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- Let us consider $f_1 = \sin\varepsilon$ and $f_2 = \sin\varepsilon + \exp(-1/\varepsilon^2)$. If

$$\sin\varepsilon + \exp(-1/\varepsilon^2) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \dots$$

then $a_0 = 0$,

$$a_1 = \lim_{\varepsilon\to 0}\Big(\frac{\sin\varepsilon + \exp(-1/\varepsilon^2)}{\varepsilon}\Big) = \lim_{\varepsilon\to 0}\Big(\frac{\sin\varepsilon}{\varepsilon}\Big)$$

$$+ \lim_{\varepsilon\to 0}\Big(\frac{+\exp(-1/\varepsilon^2)}{\varepsilon}\Big) = 1 + 0 = 1$$

Uniqueness of expansions

$$a_2 = \lim_{\varepsilon \to 0} \left(\frac{\sin \varepsilon + \exp(-1/\varepsilon^2) - \varepsilon}{\varepsilon^2} \right) = 0$$

$$a_3 = \lim_{\varepsilon \to 0} \left(\frac{\sin \varepsilon + \exp(-1/\varepsilon^2) - \varepsilon}{\varepsilon^3} \right) = -\frac{1}{6}$$

$$\exp(-1/\varepsilon^2) = 0 \cdot 1 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + 0 \cdot \varepsilon^3 + \dots$$

Taylor series is asymptotic expansion

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- $f(x) = \sum_{n=0}^{N} a_n x^n + R_N$ with

$$R_N = \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(z).$$

So

$$R_N = O(x^{N+1}) \qquad x \to 0.$$

$$\lim_{x\to 0} \frac{R_N}{x^{N+1}} = \frac{1}{(N+1)!} f^{(N+1)}(0) = 0$$

The end