Asymptotics

Lesson 5

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- $f(x) = \frac{sinx}{x}$,
- $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational.

Order symbols $O(x)$ ***and*** $o(x)$

Let us see on the limits

$$\lim_{x \to 0} \sin x =$$

$$\lim_{x \to \pi/2} \cos x =$$

$$\lim_{x \to 0} \frac{x^3}{x^2 - x} =$$

$$\lim_{x \to \infty} e^{-x} =$$

Order symbols $O(x)$ and $o(x)$

They are the zero limits

$$\lim_{x \to 0} \sin x = 0$$

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$$\lim_{x \to 0} \frac{x^3}{x^2 - x} = 0$$

$$\lim_{x \to \infty} e^{-x} = 0$$

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Let us see on other group of limits

$$\lim_{x \to 0} \cos x =$$

$$\lim_{x \to 0} \frac{1+x}{1-x} =$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} =$$

$$\lim_{x \to \infty} \frac{1+x}{1-x} =$$

Order symbols $O(x)$ and $o(x)$

Finite limits

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{1+x}{1-x} = 1$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

$$\lim_{x \to \infty} \frac{1+x}{1-x} = -1$$

Order symbols $O(x)$ and $o(x)$

The last group of limits

$$\lim_{x\to 1^-} \frac{1}{1-x} =$$

$$\lim_{x\to 1^+} \frac{1}{1-x} =$$

$$\lim_{x\to 0^+} e^{1/x} =$$

$$\lim_{x\to 0^+} \ln x =$$

$$\lim_{x\to 0} \frac{1}{1-\cos x} =$$

Order symbols $O(x)$ and $o(x)$

Infinite limits

$$\lim_{x\to 1^-} \frac{1}{1-x} = +\infty$$

$$\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$$

$$\lim_{x\to 0^+} e^{1/x} = \infty$$

$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to 0} \frac{1}{1-\cos x} = \infty$$

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Order symbols $O(x)$ *and* $o(x)$

- Why we always can consider the limits as $x \to 0$ instead of $x \to x_0$?
- $\lim_{x \to x_0} f(x) = \lim_{x - x_0 \to 0} f((x - x_0) + x_0) = \lim_{y \to 0} f(y)$

Order symbols $O(x)$ ***and*** $o(x)$

Different order $x^2 = o(x)$, $x = o(\sqrt{x})$

x	0.1	0.01	0.001	0.0001
$\sqrt{x}$	0.316	0.1	0.0316	0.01
x^2	0.01	0.0001	0.000001	0.00000001

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The same order $\sin x = O(x)$

x	0.1	0.01	0.001	0.0001
$\sin x$	0.099833	0.009999	0.001000	0.000100

Order symbols $O(x)$ ***and*** $o(x)$

If $\lim_{x \to x_0} \frac{f(x)}{g(x)} = C \neq 0$, then $f(x) = O(g(x))$ and $g(x)$ is called the *gauge* function.

The typical guage functions are powers of x, but not unique.

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If $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$, then $f(x) = o(g(x))$

$\sin x = O(x)$ as $x \to 0$ since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and

$\sin x = o(\sqrt{x})$ as $x \to 0$ since $\lim_{x \to 0} \frac{\sin x}{\sqrt{x}} = 0$

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- $\frac{x}{1+x^2} = O(1/x)$ as $x \to \infty$

L'Hospital rule

If $f(x_0) = g(x_0) = 0$ or (∞) and the limit $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then

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Consider examples

$$e^x - 1 = O(x) \text{ as } x \to 0$$

$$\sin x - x = O(x^3) \text{ as } x \to 0$$

$$\cot x = O(1/x) \text{ as } x \to 0$$

Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$
$$+ \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N + R_N$$

with

$$R_N = \frac{f^{(N+1)}(z)}{(N+1)!}(x - x_0)^{N+1} \quad z \in (x_0, x) \text{ or } (x, x_0).$$

The formula is valid if the derivative $f^{(N+1)}(z)$ exists and continuous in a neighborhood of x_0. Then

Taylor's formula

$$\lim_{x\to x_0} \frac{R_N}{(x-x_0)^{N+1}} = f^{(N+1)}(x_0)\frac{1}{(N+1)!} \quad \Rightarrow$$

$$R_N = O((x-x_0)^{N+1}) \text{ as } x \to x_0$$

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$f(x) = O(g(x))$ as $x \to x_0$ means that

$$K_1|g(x)| \le |f(x)| \le K_2|g(x)|, \qquad x \sim x_0$$

Taylor's formulas near 0

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7) \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + O(x^4)$$

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- $\frac{O(x^n)}{O(x^m)} = O(x^{n-m}), \qquad x \sim 0, \infty$
- $\big(O(x^n)\big)^p = O(x^{np}), \qquad x \sim 0, \infty$

Functions of small quantities

We have for n > 0

$$e^{O(x^n)} = 1 + O(x^n)$$

$$\ln(1 + O(x^n)) = O(x^n)$$

$$\sin(O(x^n)) = O(x^n) \quad \cos(O(x^n)) = 1 - O(x^{2n})$$

$$\tan(O(x^n)) = O(x^n)$$

$$(1 + O(x^n))^p = 1 + O(x^n)$$

Behavior of exponential function

Let $x \sim \infty$, then

$$\lim_{x\to\infty} \frac{e^{-x}}{1/x^N} = \lim_{x\to\infty} \frac{x^N}{e^{-x}}$$

$$= \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x^N} + \frac{1}{x^{N-1}} + \frac{1}{2!x^{2-N}} + \ldots + \frac{1}{N!} + \frac{x}{(N+1)!} + \frac{x^2}{(N+2)!} + \ldots\right.}$$

$$= 0$$

We conclude that

$$e^{-x} = o\Big(\frac{1}{x^N}\Big), \qquad x \to \infty$$

or $e^{-1/y} = o(y^N) \quad y \to 0, \qquad N \mapsto p \in \mathbb{R}^+$

Behavior of exponential function

We conclude

$$\lim_{x \to \infty} \frac{x^p}{e^x} = 0 \quad \Rightarrow \quad x^p = o(e^x) \;\; x \to +\infty \;\; \forall p \in \mathbb{R}$$

Behavior of logarithmic function

Put $e^x = \left(\frac{y}{p}\right)^p$ in $x^p = o(e^x)$. If $x \to +\infty$ then $y \to \infty$ for $p > 0$. Since $x = p \ln\left(\frac{y}{p}\right)$ we have

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or $\ln x = o\left(\frac{1}{x^p}\right) \quad x \to 0^+, \quad p > 0$

Behavior of logarithmic and exponential functions

$$e^{-x}x^p \to 0 \qquad x \to \infty \quad p \in \mathbb{R}$$

$$\frac{e^x}{x^p} \to \infty \qquad x \to \infty \quad p \in \mathbb{R}$$

$$x^p \ln x \to 0 \qquad x \to 0^+ \quad p \in \mathbb{R}^+$$

$$\frac{\ln x}{x^p} \to 0 \qquad x \to \infty \quad p \in \mathbb{R}^+$$

The end