Truncated rule

Lesson 4

Optimum truncated rule

$y = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + R_N, \quad |R_N| \leq N!/x^{N+1}$. Let us estimate the approximation of $y(4)$. We have

$$|R_N| \leq N!/x^{N+1} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots}{4 \times 4 \times 4 \times 4 \times 4 \times 4 \ldots}$$

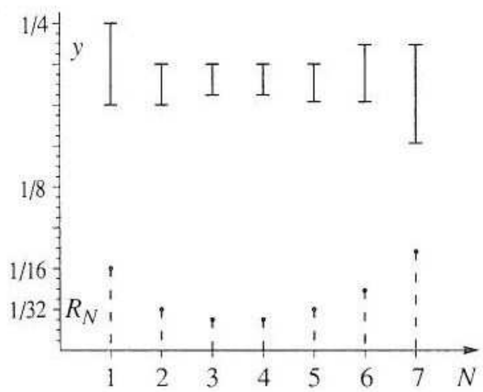

Fig. 1.2 The variation of the bound for $|R_N|$ ---- and the estimate for y |—|

$$y_{4\,terms} = 0.1953 < y(4) < 0.2187 = y_{3\,terms}$$

Optimum truncated rule

For $y(6)$ and $y(10)$ we have

$$|R_N| \leq \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\ldots}{6\cdot 6\cdot 6\cdot 6\cdot 6\cdot 6\cdot 6\ldots}$$

$$y_{6\,terms} = 0.1440 < y(6) < 0.1466 = y_{5\,terms}$$

$$|R_N| \leq \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\ldots}{10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\ldots}$$

$$y_{10\,terms} = 0.09154 < y(10) < 0.09158 = y_{9\,terms}$$

R_N is bounded by the first neglected term in the expansion and tends to zero faster then last term in the truncated series as $x \to \infty$ (asymptotic expansion).

Optimum truncated rule

Table 1.1

x	y_{accurate}	$y_{\text{estimate}} \pm$ Error bound $y_e \pm R_e$	R_e/y_e
4	0.2063456	0.2070363 ± 0.0117238	0.056627
6	0.1452676	0.1453189 ± 0.0012860	0.008853
10	0.0915633	0.0915638 ± 0.0000182	0.000199

One more example

$f(x) = \int_0^\infty \frac{e^{-xt}}{\sqrt{1+t}}\, dt, \qquad x \sim \infty.$

$$f(x) = \frac{-e^{-xt}}{x(1+t)^{1/2}}\Big|_0^\infty - \frac{1}{2x}\int_0^\infty \frac{-e^{-xt}}{(1+t)^{3/2}}\, dt$$

$$= \frac{1}{x} - \frac{1}{2x}\Big(\frac{-e^{-xt}}{x(1+t)^{3/2}}\Big|_0^\infty\Big) - \frac{3\cdot 1}{2\cdot 2}\frac{1}{x^2}\int_0^\infty \frac{-e^{-xt}}{(1+t)^{5/2}}\, dt$$

$$+\frac{1}{x} - \frac{1}{2x^2} + \frac{3\cdot 1}{2\cdot 2}\frac{1}{x^3} - \frac{5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot \ldots \cdot 2}\frac{1}{x^4}$$

$$+\ldots(-1)^{N-1}\frac{\big(2(N+1)-1\big)\cdot\ldots 5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot\cdot\ldots\cdot 2}\frac{1}{x^N} + R_N$$

One more example

with

$R_N = (-1)^{N-1} \frac{\big(2(N+1)-1\big)\cdot\ldots 5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot\cdot\ldots\cdot 2} \frac{1}{x^N} \int_0^\infty \frac{-e^{-xt}}{(1+t)^{(2(N+1)+1)/2}}\, dt,$

since $\frac{1}{1+t} < 1$ and $\int_0^\infty e^{-xt}\, dt = 1/x$ we have

$$|R_N| \leq \frac{\big(2(N+1)-1\big)\cdot\ldots 5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot\cdot\ldots\cdot 2} \frac{1}{x^N}.$$

Comparing with last non-neglecting term in the truncated series

$$L = (-1)^{N-2} \frac{\big(2(N)-1\big)\cdot\ldots 5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot\cdot\ldots\cdot 2} \frac{1}{x^{N-1}}$$

we see that $R_N \to \infty$ faster then $L \to \infty$ as $x \to \infty$.

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- We look for the asymptotic expansion for $\mathrm{erfc}(x)$

The error function

$$\operatorname{erfc}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-t^2}\,dt = \frac{2}{\pi^{1/2}} \int_x^\infty \frac{-2te^{-t^2}}{-2t}\,dt$$

$$= \frac{2}{\pi^{1/2}} \Big(\frac{-e^{-t^2}}{-2t} |_x^\infty - \int_x^\infty \frac{-e^{-t^2}}{2t^2}\,dt \Big)$$

$$= \frac{2}{\pi^{1/2}} \Big(\frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty \frac{-2te^{-t^2}}{2t^3}\,dt \Big).$$

Repeating integration by parts we get

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{\pi^{1/2}}\Big(\frac{1}{x} - \frac{1}{2x^3} + \frac{1\cdot 3}{2^2x^5} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \dots$$
$$+ (-1)^{N-1}\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2N-3)}{2^{N-1}x^{2N-1}}\Big) + R_N.$$

Does the series converge?

$$R_N = \frac{(-1)^{N-1}}{\sqrt{\pi}}\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2N-1)}{2^{N-1}}\int_x^\infty \frac{e^{-t^2}}{t^{2N}}\,dt.$$

We can estimate R_N as follows

The error function

$$\int_x^\infty \frac{e^{-t^2}}{t^{2N}}\,dt = \int_x^\infty \frac{2te^{-t^2}}{2t^{2N+1}}\,dt$$
$$< \int_x^\infty \frac{2te^{-t^2}}{2x^{2N+1}}\,dt = \frac{e^{-x^2}}{2x^{2N+1}}\,dt.$$

$|R_N| < \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2N-1)}{\sqrt{\pi}2^N x^{2N+1}} e^{-x^2}$. It is asymptotic expansion since $L = (-1)^{N-2}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2N-5)}{2^{N-2}x^{2N-3}}e^{-x^2}$.

The error function

$$\mathrm{erfc}(2) = \frac{e^{-4}}{\sqrt{\pi}}\Big(\frac{1}{2} - \frac{1}{2 \cdot 2^3} + \frac{1 \cdot 3}{2^2 \cdot 2^5} - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 2^7} + \tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 2^9} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^5 \cdot 2^{11}} + \dots\Big) \quad (1)$$

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the smallest term is $\frac{1\cdot3\cdot5\cdot7}{2^4\cdot2^9}$. The following terms begin to increase in magnitude.

$$0.004612 < \mathrm{erfc}(2) < 0.004744 \text{ or}$$

$$\mathrm{erfc}(2) = 0.004678 \pm 0.000066.$$

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Comparison with a convergent series

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$1 - \frac{2}{\pi^{1/2}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right) dt$$

$$= 1 - \frac{2}{\pi^{1/2}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots\right)$$

$$\mathrm{erfc}(2) = 1 - \frac{2}{\pi^{1/2}} \left(2 - \frac{2^3}{3} + \frac{2^5}{5 \cdot 2!} - \frac{2^7}{7 \cdot 3!} + \dots\right).$$

Why the series converges?

Comparison with a convergent series

Table 1.2

Number of terms in truncated series	Resultant sum of terms
4	1.580309
8	0.157841
12	0.007130
16	0.004689
20	0.004678

The end