Initial value problem

Lesson 3

Example of nonlinear differential equation

- $\frac{df}{dt} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, with initial condition $f(0) = 1$.

Example of nonlinear differential equation

- $\frac{df}{dt} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, with initial condition $f(0) = 1$.
- $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \varepsilon^3 f_3(t) + \dots$

Example of nonlinear differential equation

- $\frac{df}{dt} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, with initial condition $f(0) = 1$.
- $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \varepsilon^3 f_3(t) + \ldots$
-

$$\begin{aligned}\frac{df_0}{dt} + \varepsilon \frac{df_1}{dt} + \varepsilon^2 \frac{df_2}{dt} + \ldots + f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots \\ = \varepsilon(f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots)^2.\end{aligned}$$

Example of nonlinear differential equation

- $\frac{df}{dt} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, with initial condition $f(0) = 1$.
- $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \varepsilon^3 f_3(t) + \ldots$
- $$\frac{df_0}{dt} + \varepsilon \frac{df_1}{dt} + \varepsilon^2 \frac{df_2}{dt} + \ldots + f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots$$
  $$= \varepsilon(f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots)^2.$$
- with the initial condition
  $$f_0(0) + \varepsilon f_1(0) + \varepsilon^2 f_2(0) + \ldots = 1.$$

Example of nonlinear differential equation

$$\varepsilon^0 : \quad \frac{df_0}{dt} + f_0 = 0, \quad f_0(0) = 1 \Rightarrow f_0 = Ae^{-t} = e^{-t},$$

$$\varepsilon^1 : \quad \frac{df_1}{dt} + f_1 = f_0^2, \quad f_1(0) = 0 \Rightarrow f_1 = e^{-t} - e^{-2t},$$

$$\varepsilon^2 : \quad \frac{df_2}{dt} + f_2 = 2f_0 f_1, \quad f_2(0) = 0 \Rightarrow f_2 = e^{-t} - 2e^{-2t} + e^{-3t}$$

Example of nonlinear differential equation

$\frac{df_1}{dt} + f_1 = e^{-2t}$, $\quad f_1(0) = 0$.

The solution of the homogeneous equation is Ae^{-t},

the particular solution is $-e^{-2t}$. Then

$$f_1 = Ae^{-t} - e^{-2t}, \qquad A = 1,$$

$$f_1 = e^{-t} - e^{-2t}$$

Example of nonlinear differential equation

$\frac{df_2}{dt} + f_2 = 2e^{-t}(e^{-t} - e^{-2t}), \quad f_2(0) = 0.$

The solution of the homogeneous equation is Ae^{-t},

the particular solution has the form $\alpha e^{-2t} + \beta e^{-3t}$. Then

$$(-2\alpha + \alpha)e^{-2t} + (-3\beta + \beta)e^{-3t} = 2e^{-2t} - 2e^{-3t}, \Rightarrow$$

$$\alpha = -2, \quad \beta = 1$$

$$f_2 = Ae^{-t} - 2e^{-2t} + e^{-3t} \text{ with } A = 1.$$

Uniformly and ununiformly valid expansion

- The expansion is uniformly valid for all t

$$f(\varepsilon, t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots$$

The expansion is ununiformly valid for all t

$$v(\varepsilon, t) = 1 - t + \varepsilon(\frac{t^2}{2} - t) + \varepsilon^2(\frac{t^2}{2} - \frac{t^3}{6}) + \dots$$

Uniformly and ununiformly valid expansion

- The expansion is uniformly valid for all t

$$f(\varepsilon, t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots$$

The expansion is ununiformly valid for all t

$$v(\varepsilon, t) = 1 - t + \varepsilon(\frac{t^2}{2} - t) + \varepsilon^2(\frac{t^2}{2} - \frac{t^3}{6}) + \dots$$

- Let us compare the perturbation expansion with the exact solution of $\frac{df}{dt} + f = \varepsilon f^2$.

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$
- $\int \left(\frac{-1}{f} + \frac{\varepsilon}{\varepsilon f - 1} \right) df = \int dt$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$
- $\int \left(\frac{-1}{f} + \frac{\varepsilon}{\varepsilon f - 1} \right) df = \int dt$
- $-\ln|f| + \ln|\varepsilon f - 1| = t + c$.

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$
- $\int \left(\frac{-1}{f} + \frac{\varepsilon}{\varepsilon f - 1} \right) df = \int dt$
- $-\ln|f| + \ln|\varepsilon f - 1| = t + c$.
- $\ln\left|\frac{\varepsilon f - 1}{f}\right| = t + c$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$
- $\int \left(\frac{-1}{f} + \frac{\varepsilon}{\varepsilon f - 1} \right) df = \int dt$
- $-\ln|f| + \ln|\varepsilon f - 1| = t + c$.
- $\ln\left|\frac{\varepsilon f - 1}{f}\right| = t + c$
- $\frac{\varepsilon f - 1}{f} = Ae^t, \qquad A = \varepsilon - 1$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $\frac{df}{dt} = f(\varepsilon f - 1)$
- $\int \frac{df}{f(\varepsilon f - 1)} = \int dt$
- $\int \left(\frac{-1}{f} + \frac{\varepsilon}{\varepsilon f - 1} \right) df = \int dt$
- $-\ln|f| + \ln|\varepsilon f - 1| = t + c$.
- $\ln \left| \frac{\varepsilon f - 1}{f} \right| = t + c$
- $\frac{\varepsilon f - 1}{f} = Ae^t, \qquad A = \varepsilon - 1$
- $f = \frac{e^{-t}}{1 - \varepsilon(1 - e^{-t})}$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $f = \frac{e^{-t}}{1-\varepsilon(1-e^{-t})}$, $\qquad \frac{1}{1-x} = 1 + x + x^2 + o(x^2)$

  with $x = \varepsilon(1 - e^{-t})$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $f = \frac{e^{-t}}{1-\varepsilon(1-e^{-t})}$, $\qquad \frac{1}{1-x} = 1 + x + x^2 + o(x^2)$

  with $x = \varepsilon(1 - e^{-t})$
- $f = e^{-t}\big(1 + \varepsilon(1 - e^{-t}) + \varepsilon^2(1 - e^{-t})^2 + \dots\big)$

Exact solution of $\frac{df}{dt} + f = \varepsilon f^2$

- $f = \frac{e^{-t}}{1-\varepsilon(1-e^{-t})}$, $\qquad \frac{1}{1-x} = 1 + x + x^2 + o(x^2)$

  with $x = \varepsilon(1 - e^{-t})$
- $f = e^{-t}\big(1 + \varepsilon(1 - e^{-t}) + \varepsilon^2(1 - e^{-t})^2 + \dots\big)$
- $f(\varepsilon, t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots$

One more example

- $\frac{df}{dt} + f = \varepsilon t f^2$, $0 < \varepsilon \ll 1$, with $f(0) = 1$.

  This equation is not possible to solve by the separation of variables. We seek the expansion $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + o(\varepsilon)$.

One more example

- $\frac{df}{dt} + f = \varepsilon t f^2$, $0 < \varepsilon \ll 1$, with $f(0) = 1$.

  This equation is not possible to solve by the separation of variables. We seek the expansion $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + o(\varepsilon)$.

- $\frac{df_0}{dt} + \varepsilon \frac{df_1}{dt} + f_0 + \varepsilon f_1 + o(\varepsilon) = \varepsilon t f_0^2 + o(\varepsilon)$ with initial condition $f_0(0) + \varepsilon f_1(0) + o(\varepsilon) = 1$.

One more example

- $\frac{df}{dt} + f = \varepsilon t f^2$, $0 < \varepsilon \ll 1$, with $f(0) = 1$.

  This equation is not possible to solve by the separation of variables. We seek the expansion $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + o(\varepsilon)$.

- $\frac{df_0}{dt} + \varepsilon \frac{df_1}{dt} + f_0 + \varepsilon f_1 + o(\varepsilon) = \varepsilon t f_0^2 + o(\varepsilon)$ with initial condition $f_0(0) + \varepsilon f_1(0) + o(\varepsilon) = 1$.

- $$\varepsilon^0 : \ \frac{df_0}{dt} + f_0 = 0, \ f_0(0) = 1 \Rightarrow f_0 = e^{-t},$$
  $$\varepsilon^1 : \ \frac{df_1}{dt} + f_1 = t f_0^2, \ f_1(0) = 0 \Rightarrow f_1 = e^{-t} - e^{-2t} - te^{-2t}$$

One more example

- $\frac{df_1}{dt} + f_1 = tf_0^2 = te^{-2t}$. We multiply by integrating factor e^t:

$$\frac{d}{dt}(e^t f_1) = te^{-t} \qquad (\frac{df_1}{dt}e^t + e^t f_1 = te^{-t}).$$

One more example

- $\frac{df_1}{dt} + f_1 = tf_0^2 = te^{-2t}$. We multiply by integrating factor e^t:

$$\frac{d}{dt}(e^t f_1) = te^{-t} \qquad (\frac{df_1}{dt}e^t + e^t f_1 = te^{-t}).$$

- $e^t f_1 = -te^{-t} - e^{-t} + c$ with $c = 1$,

One more example

- $\frac{df_1}{dt} + f_1 = tf_0^2 = te^{-2t}$. We multiply by integrating factor e^t:

$$\frac{d}{dt}(e^t f_1) = te^{-t} \qquad (\frac{df_1}{dt}e^t + e^t f_1 = te^{-t}).$$

- $e^t f_1 = -te^{-t} - e^{-t} + c$ with $c = 1$,
- $f_1 = -te^{-2t} - e^{-2t} + e^{-t}$

One more example

- $\frac{df_1}{dt} + f_1 = tf_0^2 = te^{-2t}$. We multiply by integrating factor e^t:

$$\frac{d}{dt}(e^t f_1) = te^{-t} \qquad (\frac{df_1}{dt}e^t + e^t f_1 = te^{-t}).$$

- $e^t f_1 = -te^{-t} - e^{-t} + c$ with $c = 1$,
- $f_1 = -te^{-2t} - e^{-2t} + e^{-t}$
- The expansion is found

$$f = e^{-t} + \varepsilon(e^{-t} - e^{-2t} - te^{-2t})$$

Example with second derivative

- $\frac{d^2 f}{dt^2} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, $f(0) = 1$, $\frac{df}{dt}(0) = -1$,

Example with second derivative

- $\frac{d^2 f}{dt^2} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, $f(0) = 1$, $\frac{df}{dt}(0) = -1$,
- $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + o(\varepsilon)$.

Example with second derivative

- $\frac{d^2 f}{dt^2} + f = \varepsilon f^2$, $0 < \varepsilon \ll 1$, $f(0) = 1$, $\frac{df}{dt}(0) = -1$,
- $f(\varepsilon, t) = f_0(t) + \varepsilon f_1(t) + o(\varepsilon)$.
-

$$\varepsilon^0: \ \frac{d^2 f_0}{dt^2} + f_0 = 0, \ f_0(0) = 1, \frac{df_0}{dt}(0) = -1, \Rightarrow$$

$$f_0 = A\cos t + B\sin t \quad \text{with} \quad A = 1, \ B = -1,$$

$$\varepsilon^1: \ \frac{d^2 f_1}{dt^2} + f_1 = (\cos t - \sin t)^2 = 1 - \sin 2t,$$

$$f_1(0) = 0, \frac{df_1}{dt}(0) = 0, \Rightarrow$$

Example with second derivative

- The solution of the homogeneous equation is $A\cos t + B\sin t$ and a particular solution is of the form $\alpha + \beta \sin 2t$.

Example with second derivative

- The solution of the homogeneous equation is $A\cos t + B\sin t$ and a particular solution is of the form $\alpha + \beta\sin 2t$.
- $\alpha - 4\beta\sin 2t + \beta\sin 2t = 1 - \sin 2t \;\Rightarrow\; \alpha = 1, \beta = 1/3$

Example with second derivative

- The solution of the homogeneous equation is $A\cos t + B\sin t$ and a particular solution is of the form $\alpha + \beta\sin 2t$.
- $\alpha - 4\beta\sin 2t + \beta\sin 2t = 1 - \sin 2t \;\Rightarrow\; \alpha = 1, \beta = 1/3$
- $f_1 = A\cos t + B\sin t + 1 + \sin 2t/3$ with $A = -1$, $B = -2/3$.

Example with second derivative

- The solution of the homogeneous equation is $A\cos t + B\sin t$ and a particular solution is of the form $\alpha + \beta\sin 2t$.
- $\alpha - 4\beta\sin 2t + \beta\sin 2t = 1 - \sin 2t \;\Rightarrow\; \alpha = 1, \beta = 1/3$
- $f_1 = A\cos t + B\sin t + 1 + \sin 2t/3$ with $A = -1$, $B = -2/3$.
- The expansion is found

$$f = \cos t - \sin t + \varepsilon\Big(1 - \cos t - \frac{2}{3}\sin t + \frac{1}{3}\sin 2t\Big) + o(\varepsilon).$$

Estimate the value of the integral $f(x) = \int_0^x e^{-s^3}\, ds$. We shall use the Taylor series at 0 (Maclauren series)

$$f(x) = \int_0^x \left(1 - s^3 + \frac{s^6}{2} - \frac{s^9}{6} + o(s^9)\right) ds.$$

Then

$$f(x) = x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + o(x^{10})$$

for small values of x.

Expansions involving the independent series

Estimate the value of the integral $f(x) = \int_0^x e^{-s^3}\,ds$. We shall use the Taylor series at 0 (Maclauren series)

$$f(x) = \int_0^x \left(1 - s^3 + \frac{s^6}{2} - \frac{s^9}{6} + o(s^9)\right) ds.$$

Then

$$f(x) = x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + o(x^{10})$$

for small values of x.

What should we do if x is large?

Expansions involving the independent series

$$f(x) = \frac{1+x}{1-2x} = -\frac{\frac{1}{2}+\frac{1}{2x}}{1-\frac{1}{2x}}, \qquad (1-\delta)^{-1} = 1+\delta+\delta^2+\dots$$

Expansions involving the independent series

$$f(x) = \frac{1+x}{1-2x} = -\frac{\frac{1}{2}+\frac{1}{2x}}{1-\frac{1}{2x}}, \qquad (1-\delta)^{-1} = 1+\delta+\delta^2+\dots$$

$$\begin{aligned} f(x) &= -\frac{\frac{1}{2}+\frac{1}{2x}}{1-\frac{1}{2x}} = -\left(\frac{1}{2}+\frac{1}{2x}\right)\left(1+\frac{1}{2x}+\frac{1}{4x^2}+\dots\right) \\ &= -\frac{1}{2}-\frac{1}{4x}-\frac{1}{8x^2}+\dots-\frac{1}{2x}-\frac{1}{4x^2}+\dots \\ &= -\frac{1}{2}-\frac{3}{4x}-\frac{3}{8x^2}+\dots \to -\frac{1}{2} \text{ as } x\to\infty \end{aligned}$$

$$f(10) \simeq -0.57875 \qquad f(10) = -\frac{11}{19} = 0.57894.$$

Solutions of ODE in series

- $(1 - x^2)\frac{dy}{dx} = 2y$

Solutions of ODE in series

- $(1-x^2)\frac{dy}{dx} = 2y$
- $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \quad x \sim 0$.

Solutions of ODE in series

- $(1-x^2)\frac{dy}{dx} = 2y$
- $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \quad x \sim 0$.
-

$$\begin{aligned}
x^0 &: \quad a_1 = 2a_0, \\
x^1 &: \quad 2a_2 = 2a_1, \quad \Rightarrow a_2 = 2a_0, \\
x^2 &: \quad 3a_3 - a_1 = 2a_2 \quad \Rightarrow a_3 = 2a_0, \\
x^3 &: \quad 4a_4 - 2a_2 = 2a_3 \quad \Rightarrow a_4 = 2a_0.
\end{aligned}$$

Solutions of ODE in series

- $(1 - x^2)\frac{dy}{dx} = 2y$
- $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \quad x \sim 0.$
- $$\begin{aligned} x^0 &: \quad a_1 = 2a_0, \\ x^1 &: \quad 2a_2 = 2a_1, \quad \Rightarrow a_2 = 2a_0, \\ x^2 &: \quad 3a_3 - a_1 = 2a_2 \quad \Rightarrow a_3 = 2a_0, \\ x^3 &: \quad 4a_4 - 2a_2 = 2a_3 \quad \Rightarrow a_4 = 2a_0. \end{aligned}$$
- $y = a_0\big(1 + 2(x + x^2 + x^3 + x^4 + \ldots)\big)$, where a_0 is determined from the initial value $y(0) = a_0$.

Solutions of ODE in series

- $(1-x^2)\frac{dy}{dx} = 2y$

Solutions of ODE in series

- $(1 - x^2)\frac{dy}{dx} = 2y$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots, \quad x \sim \infty$.

Solutions of ODE in series

- $(1-x^2)\frac{dy}{dx} = 2y$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots, \quad x \sim \infty$.
-

$$
\begin{aligned}
x^0: &\quad b_1 = 2b_0, \\
x^{-1}: &\quad 2b_2 = 2b_1, \quad \Rightarrow b_2 = 2b_0, \\
x^{-2}: &\quad 3b_3 - b_1 = 2b_2 \quad \Rightarrow b_3 = 2b_0, \\
x^{-3}: &\quad 4b_4 - 2b_2 = 2b_3 \quad \Rightarrow b_4 = 2b_0.
\end{aligned}
$$

Solutions of ODE in series

- $(1-x^2)\frac{dy}{dx} = 2y$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots, \qquad x \sim \infty$.
- $$\begin{aligned} x^0: &\quad b_1 = 2b_0, \\ x^{-1}: &\quad 2b_2 = 2b_1, \quad \Rightarrow b_2 = 2b_0, \\ x^{-2}: &\quad 3b_3 - b_1 = 2b_2 \quad \Rightarrow b_3 = 2b_0, \\ x^{-3}: &\quad 4b_4 - 2b_2 = 2b_3 \quad \Rightarrow b_4 = 2b_0. \end{aligned}$$
- $y = b_0\big(1 + 2(x^{-1} + x^{-2} + x^{-3} + x^{-4} + \ldots)\big)$, where b_0 is determined from the value $y(\infty) = b_0$.

Convergent and divergent series

- $\sum_{n=1}^{\infty} u_n$ is converges if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

Convergent and divergent series

- $\sum_{n=1}^{\infty} u_n$ is converges if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$
- $\lim_{n\to\infty}\left|\frac{2a_0x^{n+1}}{2a_0x^n}\right| < 1$ converges for $|x| < 1$

Convergent and divergent series

- $\sum_{n=1}^{\infty} u_n$ is converges if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$
- $\lim_{n\to\infty}\left|\frac{2a_0 x^{n+1}}{2a_0 x^n}\right| < 1$ converges for $|x| < 1$
- $\lim_{n\to\infty}\left|\frac{2b_0/x^{n+1}}{2b_0/x^n}\right| < 1$ converges for $|x| > 1$.

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \qquad y(\infty) = 0$

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \qquad y(\infty) = 0$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots \qquad x \sim \infty, \quad b_0 = 0$

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \qquad y(\infty) = 0$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots \qquad x \sim \infty, \quad b_0 = 0$
-

$$
\begin{aligned}
x^{-1} &: \quad -b_1 = -1 \quad \Rightarrow b_1 = 1, \\
x^{-2} &: \quad -b_1 - b_2 = 0 \quad \Rightarrow b_2 = -1, \\
x^{-3} &: \quad -2b_2 - b_3 = 0 \quad \Rightarrow b_3 = 2!, \\
x^{-4} &: \quad -3b_3 - b_4 = 0 \quad \Rightarrow b_4 = -3!.
\end{aligned}
$$

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \quad y(\infty) = 0$
- $y = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots \quad x \sim \infty, \;\; b_0 = 0$
-

$$
\begin{aligned}
x^{-1} &: \quad -b_1 = -1 \quad \Rightarrow b_1 = 1, \\
x^{-2} &: \quad -b_1 - b_2 = 0 \quad \Rightarrow b_2 = -1, \\
x^{-3} &: \quad -2b_2 - b_3 = 0 \quad \Rightarrow b_3 = 2!, \\
x^{-4} &: \quad -3b_3 - b_4 = 0 \quad \Rightarrow b_4 = -3!.
\end{aligned}
$$

- $b_n = (-1)^{n-1}(n-1)!$ and

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \qquad y(\infty) = 0$

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \quad y(\infty) = 0$
- $y = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{n-1}(n-1)!}{x^n} + \ldots$

Divergent solution

- $\frac{dy}{dx} - y = -\frac{1}{x}, \qquad y(\infty) = 0$
- $y = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{n-1}(n-1)!}{x^n} + \ldots$
- $\lim_{n\to\infty} \left| \frac{(-1)^n n!}{x^{n+1}} \frac{x^n}{(-1)^{n-1}(n-1)!} \right| < 1 \quad \Rightarrow$

$$\frac{1}{|x|} < \frac{1}{\lim_{n\to\infty} n} = 0.$$

The series diverges $\forall x$. Nevertheless, the first few terms provide a good approximation.

Divergent solution

$$y = S_N + R_N = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \ldots + \frac{(-1)^{N-1}(N-1)!}{x^N} + R_N$$

Divergent solution

$y = S_N + R_N = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \ldots + \frac{(-1)^{N-1}(N-1)!}{x^N} + R_N$
Multiplying by the integrating factor e^{-x}, yields

$$\frac{d}{dx}(e^{-x}y) = -\frac{e^{-x}}{x}$$

$$e^{-s}y(s)|_x^{\infty} = e^{-\infty}y(\infty) - e^{-x}y(x) = -\int_x^{\infty} \frac{e^{-s}}{s}\, ds$$

$$y(x) = e^x \int_x^{\infty} \frac{e^{-s}}{s}\, ds.$$ Integrating by parts

Divergent solution

$$y(x) = e^x\Big(- \frac{e^{-s}}{s}|_x^\infty - \int_x^\infty \frac{e^{-s}}{s^2}\,ds\Big) = \frac{1}{x} - e^x \int_x^\infty \frac{e^{-s}}{s^2}\,ds$$

$$= \frac{1}{x} - e^x\Big(-\frac{e^{-s}}{s^2}|_x^\infty - \int_x^\infty 2\frac{e^{-s}}{s^2}\,ds\Big) = \frac{1}{x} - \frac{1}{x^2} + 2e^x \int_x^\infty \frac{e^{-s}}{s^3}\,ds \ldots$$

$$y = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{N-1}(N-1)!}{x^N}$$

$$+(-1)^N N! e^x \int_x^\infty \frac{e^{-s}}{s^{N+1}}\,ds$$

Divergent solution

$$y(x) = e^x\Big(-\frac{e^{-s}}{s}|_x^\infty - \int_x^\infty \frac{e^{-s}}{s^2}\,ds\Big) = \frac{1}{x} - e^x \int_x^\infty \frac{e^{-s}}{s^2}\,ds$$

$$= \frac{1}{x} - e^x\Big(-\frac{e^{-s}}{s^2}|_x^\infty - \int_x^\infty 2\frac{e^{-s}}{s^2}\,ds\Big) = \frac{1}{x} - \frac{1}{x^2} + 2e^x \int_x^\infty \frac{e^{-s}}{s^3}\,ds \ldots$$

$$y = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{N-1}(N-1)!}{x^N}$$

$$+(-1)^N N! e^x \int_x^\infty \frac{e^{-s}}{s^{N+1}}\,ds$$

where R_N is the last term.

Asymptotic series

$$y = e^x \int_x^\infty \frac{e^{-s}}{s}\, ds < e^x \int_x^\infty \frac{e^{-s}}{x}\, ds = \frac{e^x}{x} e^{-s}|_x^\infty = \frac{1}{x}.$$

The value of y is bounded: $0 < y < 1/x$. Let us fix N and let $x \to \infty$

$$|R_N| = N! e^x \int_x^\infty \frac{e^{-s}}{s^{N+1}}\, ds \le N! e^x \int_x^\infty \frac{e^{-s}}{x^{N+1}}\, ds = \frac{N!}{x^{N+1}}$$

Asymptotic series

- DOES THE LIMIT OF R_N AS $N \to \infty$ EQUAL ZERO FOR x FIXED?

Asymptotic series

- DOES THE LIMIT OF R_N AS $N \to \infty$ EQUAL ZERO FOR x FIXED?
- Does the limit of R_N as $x \to \infty$ equal zero for N fixed?

Asymptotic series

- DOES THE LIMIT OF R_N AS $N \to \infty$ EQUAL ZERO FOR x FIXED?
- Does the limit of R_N as $x \to \infty$ equal zero for N fixed?
- If it is so, does R_N approach zero faster then the terms in the truncated series?

Asymptotic series

- DOES THE LIMIT OF R_N AS $N \to \infty$ EQUAL ZERO FOR x FIXED?
- Does the limit of R_N as $x \to \infty$ equal zero for N fixed?
- If it is so, does R_N approach zero faster then the terms in the truncated series?
- The series satisfying the last property is called *asymptotic series*

The end