Principal of least degeneracy

Higher order matching and nonlinear equations

Lesson 16

A boundary layer of thickness $O(\sqrt{\varepsilon})$

Let us consider the example $\varepsilon \frac{d^2 f}{dx^2} + x^2 \frac{df}{dx} - f = 0$ with $x \in (0,1)$, $f(0) = 1$, $f(1) = 2$. We will look for the one term expansion. Since $x^2 > 1$ on $(0,1)$ we assume that the boundary layer is at $x = 0$. Is something strange up to now? Thus

$$x^2 \frac{df_0^{out}}{dx} - f_0^{out} = 0, \qquad f_0^{out}(1) = 2.$$

Separating variables leads to

$$\int \frac{df_0^{out}}{f_0^{out}} = \int \frac{dx}{x^2} \quad \Rightarrow \quad f_0^{out} = Ce^{-1/x} = 2e^{1-1/x}.$$

Thickness $O(\sqrt{\varepsilon})$

We suppose that $s = x/\varepsilon$, then the new equation is

$$\frac{d^2 f}{ds^2} + \varepsilon^2 s^2 \frac{df}{ds} - \varepsilon f = 0, \qquad f(0) = 1$$

then

$$\frac{d^2 f_0^{in}}{ds^2} = 0 \quad \Rightarrow \quad f_0^{in} = As + 1.$$

Applying matching condition, we get

$$\lim_{x \to 0} 2e^{1-1/x} = \lim_{s \to \infty} As + 1.$$

What is wrong?

Thickness $O(\sqrt{\varepsilon})$

Let us try the stretched variable $s = x/\varepsilon^p$, then

$$\frac{df}{dx} = \frac{1}{\varepsilon^p}\frac{df}{ds}, \qquad \frac{d^2f}{dx^2} = \frac{1}{\varepsilon^{2p}}\frac{d^2f}{ds^2}.$$

$$\varepsilon^{1-2p}\frac{d^2f}{ds^2} + \varepsilon^p s^2\frac{df}{ds} - f = 0.$$

Let us try to keep the maximum number of terms in the equation. Taking $p = 1/2$ we keep the second derivative and f. Then

$$\frac{d^2f_0^{in}}{ds^2} - f_0^{in} = 0, f_0^{in}(0) = 1, \Rightarrow \; f_0^{in} = Ae^s + (1-A)e^{-s}.$$

Thickness $O(\sqrt{\varepsilon})$

Applying Prandtl's matching condition, we get

$$\lim_{x\to 0} 2e^{1-1/x} = \lim_{s\to\infty} Ae^{s} + (1-A)e^{-s}$$

that implies $A = 2$. Thus

$$f_0^{comp} = 2e^{1-1/x} + e^{-x/\sqrt{\varepsilon}}.$$

If $p < 1/2$ then $f_0^{in} = 0$ which does not satisfy the boundary condition at $s = 0$.

An interior boundary layer

$$\varepsilon\frac{d^2 f}{dx^2}+x\frac{df}{dx}+xf = 0 \text{ with } x\in(-1,1), f(-1)=e, f(1)=2e^{-1}.$$

We expect the boundary layer at $x=0$, two outer solution $f_0^{+out}=f_0^+$ and $f_0^{-out}=f_0^-$. We have

$$\frac{df_0^+}{dx}+f_0^+=0,\quad f_0^+(1)=2e^{-1}\quad\Rightarrow\quad f_0^+=2e^{-x},$$

$$\frac{df_0^-}{dx}+f_0^-=0,\quad f_0^-(-1)=e\quad\Rightarrow\quad f_0^-=e^{-x}.$$

An interior boundary layer

The stretched variable $s = x/\varepsilon^p$ leads to

$$\varepsilon^{1-2p}\frac{d^2 f}{ds^2} + s\frac{df}{ds} + \varepsilon^p s f = 0.$$

Taking $p = 1/2$, we get

$$\frac{d^2 f_0^{in}}{ds^2} + s\frac{df_0^{in}}{ds} = 0.$$

We make a substitution $w = \frac{df_0^{in}}{ds}$, then

$$\frac{dw}{ds} + sw = 0 \quad \Rightarrow \quad w = Ae^{-s^2/2} \quad \Rightarrow$$

An interior boundary layer

$$f_0^{in} = A \int_0^s e^{-t^2/2}\, dt + f_0^{in}(0).$$

We write $\int_0^s e^{-t^2/2}\, dt = \frac{1}{\sqrt{2}} \int_0^{s/\sqrt{2}} e^{-u^2}\, du = B \operatorname{erf}(s/\sqrt{2})$.
Then

$$f_0^{in} = B \operatorname{erf}(s/\sqrt{2}) + f_0^{in}(0).$$

The Prandtl's condition gives

$$\lim_{x \to 0^+} 2e^{-x} = \lim_{s \to +\infty} B \operatorname{erf}(s/\sqrt{2}) + f_0^{in}(0) \quad \Rightarrow \quad B + f_0^{in}(0) = 2,$$

$$\lim_{x \to 0^-} e^{-x} = \lim_{s \to +\infty} B \operatorname{erf}(s/\sqrt{2}) + f_0^{in}(0) \quad \Rightarrow \quad -B + f_0^{in}(0) = 1.$$

An interior boundary layer

$$B = 1/2, \quad f_0^{in}(0) = 3/2.$$

Summarizing

$$f_0^+ = 2e^{-x}, \qquad x > O(\sqrt{\varepsilon})$$
$$f_0^{in} = 1/2\,\mathrm{erf}(x/\sqrt{2\varepsilon}) + 3/2, \qquad x = O(\sqrt{\varepsilon})$$
$$f_0^- = e^{-x}, \qquad x < -O(\sqrt{\varepsilon})$$

The composition solution can not be produced by the standard way. We observe

An interior boundary layer

$f_0^{in}\Big(x > O(\sqrt{\varepsilon})\Big) = 1/2 + 3/2 +$ sm. t., since $\mathrm{erf} \sim 1$,

$f_0^{in}\Big(x < -O(\sqrt{\varepsilon})\Big) = -1/2 + 3/2 +$ sm. t., since $\mathrm{erf} \sim -1$,

We conclude that

$$f_0^{comp} = \Big(1/2\,\mathrm{erf}(x/\sqrt{2\varepsilon}) + 3/2\Big)e^{-x}.$$

An interior boundary layer

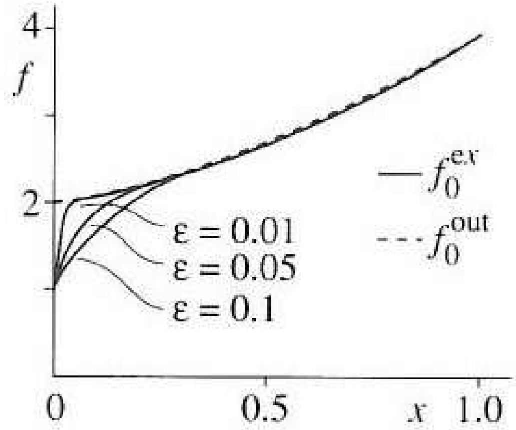

Fig. 5.1 The exact solution for various values of ε

Higher order matching

We shall work with the old example

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1, \quad x \in (0,1), f(0) = 1, f(1) = 4,$$

$$f^{ex} = x^2 + x + 2 - e^{-x/\varepsilon} + \varepsilon\Big(2(1-x) - 2e^{-x/\varepsilon}\Big)$$

Using the stretched variable $s = x/\varepsilon$ we get

Higher order matching

$$\frac{d^2 f}{ds^2} + \frac{df}{ds} = 2\varepsilon^2 s + \varepsilon, \quad f(s,\varepsilon) \sim f_0(s) + \varepsilon f_1(s) + \ldots,$$

$$\varepsilon^0 \; \frac{d^2 f_0^{in}}{ds^2} + \frac{df_0^{in}}{ds} = 0, f_0^{in}(0) = 1 \Rightarrow f_0^{in} = A + (1-A)e^{-s}$$

$$\varepsilon^1 \; \frac{d^2 f_1^{in}}{ds^2} + \frac{df_1^{in}}{ds} = 1, f_1^{in}(0) = 0 \Rightarrow f_1^{in} = B - Be^{-s} + s$$

$$\varepsilon^2 \; \frac{d^2 f_2^{in}}{ds^2} + \frac{df_2^{in}}{ds} = 2s, f_2^{in}(0) = 0 \Rightarrow f_2^{in} = C - Ce^{-s} + s^2 - 2s$$

$$\varepsilon^n \; \frac{d^2 f_n^{in}}{ds^2} + \frac{df_n^{in}}{ds} = 0, f_n^{in}(0) = 0 \Rightarrow f_n^{in} = D_n - D_n e^{-s},$$

$n = 3, 4, \ldots$

Higher order matching

$$f^{in} = A + (1-A)e^{-s} + \varepsilon(B - Be^{-s} + s)$$
$$+\varepsilon^2(C - Ce^{-s} + s^2 - 2s) + \sum_{n=3}^{\infty} \varepsilon^n D_n(1 - e^{-s}).$$

The Prandtl's condition can not be applied, because of existence of unbounded terms εs and $\varepsilon^2(s^2 - 2s)$. They represent x and $x^2 - 2\varepsilon x$. Let us write the exact solution in terms of s:

Higher order matching

$$f^{ex} = x^2 + x + 2 - e^{-x/\varepsilon} + \varepsilon\Big(2(1-x) - 2e^{-x/\varepsilon}\Big)$$

$$= \varepsilon^2 s^2 + \varepsilon s + 2 - e^{-s} + \varepsilon\big(2(1-\varepsilon s) - 2e^{-s}\big)$$

$$= 2 - e^{-s} + \varepsilon(2 - 2e^{-s} + s) + \varepsilon^2(s^2 - 2s).$$

Comparing with f^{in} we get $A = 2$, $B = 2$, $C = 0$, $D_n = 0$, $n = 3, 4, \ldots$. How we can define A, B, C, D_n without of exact solution?

Higher order matching

Let us look on the Prandtl's condition from the other point of view. Let $x = O(\varepsilon^{1/2})$, then

$$f^{in} = A + (1 - A)e^{-s} = A + o(1), \ f^{out} = x^2 + x + 2 - e^{-x/\varepsilon} = 2 + o($$

We recuperated $A = 2$. Let us do the same, but in slightly general form. We represent $s = \frac{x}{\varepsilon}$ in two variables $x = t\varepsilon^{\alpha}$ then $s = \frac{t}{\varepsilon^{1-\alpha}}$. Let us represent two term expansion in terms of new variable $t = x/\varepsilon^{\alpha}$, $\alpha \in (0, 1)$ for t fixed. We get

$$f^{out} = x^2 + x + 2 + \varepsilon 2(1 - x) = 2 + \varepsilon^{\alpha} t + \varepsilon^{2\alpha} t^2 + 2\varepsilon - 2\varepsilon^{1+\alpha} t$$

Higher order matching

Now we use $s = \frac{t}{\varepsilon^{1-\alpha}}$ and fined two term expansion of f^{in} in terms of $\frac{t}{\varepsilon^{1-\alpha}}$.

$$f^{in} = A + \varepsilon(B + s) = A + \varepsilon^{\alpha} t + \varepsilon B.$$

Matching with

$$f^{out} = 2 + \varepsilon^{\alpha} t + \varepsilon^{2\alpha} t^2 + 2\varepsilon - 2\varepsilon^{1+\alpha} t$$

we have to ask $\varepsilon^{2\alpha} = o(\varepsilon)$, or $\alpha > 1/2$. We get $A = 2$ and $B = 2$.

Higher order matching

The composition solutions is

$$f^{comp}_{2\,term} = f^{out}_{2\,term} + f^{in}_{2\,term} - f^{match}_{2\,term}$$

where $f^{comp}_{2\,term}$ is given by terms of order up to $O(\varepsilon)$. In our case it is $2 + x + 2\varepsilon$. We have

$$f^{comp} = x^2 + x + 2 + 2\varepsilon(1 - x)$$

$$+2 - e^{-x/\varepsilon} + \varepsilon(2 - 2e^{-x/\varepsilon} + x/\varepsilon)$$

$$-2 - x - 2\varepsilon = x^2 + x + 2 - e^{-x/\varepsilon} + \varepsilon(2 - 2x - 2e^{-x/\varepsilon}).$$

Nonlinear examples

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f^2 = 0, \qquad x \in (0,1), f(0) = 2, f(1) = 1/2.$$

$$\frac{df_0^{out}}{dx} + (f_0^{out})^2 = 0, \qquad f_0^{out}(1) = 1/2 \quad \Rightarrow \quad f_0^{out} = \frac{1}{x+1}.$$

$$s = \frac{x}{\varepsilon} \quad \Rightarrow \quad \frac{d^2 f^{in}}{ds^2} + \frac{df^{in}}{ds} + \varepsilon (f^{in})^2 = 0$$

$$\frac{d^2 f_0^{in}}{ds^2} + \frac{df_0^{in}}{ds} = 0 \quad \Rightarrow \quad f_0^{in} = A + (2 - A)e^{-s}$$

Nonlinear examples

The Prandtl's condition implies

$$\lim_{x\to 0}\frac{1}{x+1} = \lim_{s\to\infty} A + (2-A)e^{-s} \quad \Rightarrow \quad A = 1.$$

$$f^{comp} = \frac{1}{x+1} + e^{-x/\varepsilon}.$$

Nonlinear examples

$$\varepsilon \frac{d^2 f}{dx^2} + 2f \frac{df}{dx} - 4f = 0, \quad x \in (0,1), f(0) = 0, f(1) = 4.$$

$$2f_0^{out} \frac{df_0^{out}}{dx} - 4f_0^{out} = 0, f(1) = 4 \quad \Rightarrow \quad f_0^{out} = 2x + 2$$

$$s = \frac{x}{\varepsilon} \quad \Rightarrow \quad \frac{d^2 f_0^{in}}{ds^2} + 2f_0^{in} \frac{df_0^{in}}{ds} = 0 \Rightarrow \quad \frac{d}{ds}\left(\frac{df_0^{in}}{ds} + (f_0^{in})^2 \right) = 0$$

$$\frac{df_0^{in}}{ds} + (f_0^{in})^2 = c = a^2$$

since $f_0^{in}(0) = 0$ and $\lim_{x \to 0} f_0^{out} = 2$ and f_0^{in} increase monotonically.

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- $A = 1$, $f_0^{in} = a\frac{e^{2as}-1}{e^{2as}+1} = a \tanh as$
- $\lim_{x\to 0} f_0^{out} = \lim_{s\to\infty} f_0^{in} \quad \Rightarrow \quad a = 2$

Nonlinear examples

$$f_0^{in} = a \tanh 2s$$

$$f^{comp} = 2x + 2\tanh(2x/\varepsilon).$$

The end