Boundary layer

Model problem

Lesson 14

Matched asymptotic expansions

- The fluid velocity near the solid wall

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- The fluid velocity near the solid wall
- The velocity at the edge of a jet of fluid

Matched asymptotic expansions

- The fluid velocity near the solid wall
- The velocity at the edge of a jet of fluid
- Condensing vapor on a cool surface

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- Constructing of the outer expansion (straightforward expansion)

Matched asymptotic expansions

- The fluid velocity near the solid wall
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Matched asymptotic expansions

- The fluid velocity near the solid wall
- The velocity at the edge of a jet of fluid
- Condensing vapor on a cool surface
- Constructing of the outer expansion (straightforward expansion)
- Constructing of the inner expansion
- Matching them together obtaining matched asymptotic expansion

Model example

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1,\ \varepsilon \ll 1,\ x \in (0,1),\ f(0) = 1,\ f(1) = 4.$$

$$f(x,\varepsilon) \sim f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \dots,$$

$$\varepsilon^0 : \ \frac{df_0}{dx} = 2x + 1, \quad f_0(0) = 1, \ \ f_0(1) = 4$$

$$\varepsilon^n : \ \frac{df_n}{dx} = -\frac{d^2 f_{n-1}}{dx^2} \ \text{ for } \ n \in \mathbb{N}, \quad f_n(0) = 0, \ \ f_n(1) = 0,$$

the differential equation of the first order can not satisfy two boundary conditions.

Model example

The general solution of the first equation $\frac{df_0}{dx} = 2x + 1$ is $f_0 = x^2 + x + c$. We abandon the initial condition at $x = 0$ and use

$$f_0(1) = 4 \quad \Rightarrow \quad c = 2 \text{ and } f_0 = x^2 + x + 2$$

$$\frac{df_1}{dx} = -\frac{d^2 f_0}{dx^2} = -2 \quad \Rightarrow \quad f_1 = -2(x - 1).$$

$$f^{outer} = x^2 + x + 2 + \varepsilon 2(1 - x)$$

$$f^{outer}(0) \neq 1$$

Model example

Let us look on the exact solution of $\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1$, This is a linear equation with constant coefficients. Its exact solution is

$$f = A + Be^{-x/\varepsilon} + x^2 + x(1-2\varepsilon), \quad A = 2(1+\varepsilon), B = -(1+2\varepsilon)$$

because we neglect the term $e^{-x/\varepsilon} = o(\varepsilon^N)$. The exact solution is

$$f^{exact} = 2(1+\varepsilon) - (1+2\varepsilon)e^{-x/\varepsilon} + x^2 + x(1-2\varepsilon)$$

$$= x^2 + x + 2 - e^{-x/\varepsilon} + \varepsilon\big(2(1-x) - 2e^{-x/\varepsilon}\big)$$

The term $e^{-x/\varepsilon}$ is absent in f^{outer}.

Model example

$\varepsilon = 0.1$

$x =$	0	0.1	0.2	0.3	0.4
$e^{-x/\varepsilon} =$	e^{0}	e^{-1}	e^{-2}	e^{-3}	e^{-4}
	$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
	1.00	0.368	0.135	0.0498	0.0183

$\varepsilon = 0.01$

$x =$	0	0.01	0.02	0.03	0.04
$e^{-x/\varepsilon} =$	e^{0}	e^{-1}	e^{-2}	e^{-3}	e^{-4}
	$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
	1.00	0.368	0.135	0.0498	0.0183.

Model example

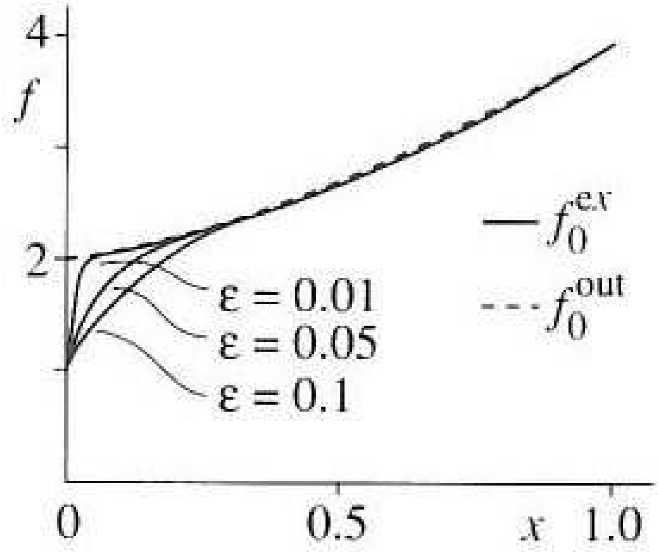

Fig. 5.1 The exact solution for various values of ε

The region near $x = 0$ is called the boundary layer.

The stretched variables and inner expansion

The gradient of f^{exact} is increase rapidly in the boundary layer.

$$f_0^{ex} = x^2 + x + 2 - e^{-x/\varepsilon}, \ \frac{df_0^{ex}}{dx} = 2x + 1 + \frac{1}{\varepsilon}e^{-x/\varepsilon},$$

$$\frac{d^2 f_0^{ex}}{dx^2} = 2 - \frac{1}{\varepsilon^2}e^{-x/\varepsilon}$$

out. BL $\quad f_0^{ex} = O(1), \ \frac{df_0^{ex}}{dx} = O(1), \ \frac{d^2 f_0^{ex}}{dx^2} = O(1),$

ins. BL $\quad f_0^{ex} = O(1), \ \frac{df_0^{ex}}{dx} = O(\frac{1}{\varepsilon}), \ \frac{d^2 f_0^{ex}}{dx^2} = O(\frac{1}{\varepsilon^2}),$

The stretched variables and inner expansion

Let us introduce the new variable $s = \frac{x}{\varepsilon}$. Then

$$\frac{df}{ds} = \frac{df}{dx}\frac{dx}{ds} = \varepsilon\frac{df}{dx}, \qquad \frac{df}{dx} = O(\frac{1}{\varepsilon}) \mapsto \frac{df}{ds} = O(1),$$

$$\frac{d^2f}{dx^2} = \varepsilon^2\frac{d^2f}{dx^2}, \qquad \frac{d^2f}{dx^2} = O(\frac{1}{\varepsilon^2}) \mapsto \frac{d^2f}{ds^2} = O(1).$$

New variable $s = \frac{x}{\varepsilon}$ is called stretched variable. The equation $\varepsilon\frac{d^2f}{dx^2} + \frac{df}{dx} = 2x + 1$ take the form

$$\frac{\varepsilon}{\varepsilon^2}\frac{d^2f}{ds^2} + \frac{1}{\varepsilon}\frac{df}{ds} = 2\varepsilon s + 1,$$

multiplying by ε we get

The stretched variables and inner expansion

$$\frac{d^2 f}{ds^2} + \frac{df}{ds} = 2\varepsilon^2 s + \varepsilon, \quad f(s,\varepsilon) \sim f_0(s) + \varepsilon f_1(s) + \dots,$$

$$\varepsilon^0 \ \frac{d^2 f_0^{in}}{ds^2} + \frac{df_0^{in}}{ds} = 0, f_0^{in}(0) = 1 \Rightarrow f_0^{in} = A + (1-A)e^{-s}$$

$$\varepsilon^1 \ \frac{d^2 f_1^{in}}{ds^2} + \frac{df_1^{in}}{ds} = 1, f_1^{in}(0) = 0 \Rightarrow f_1^{in} = B - Be^{-s} + s$$

$$\varepsilon^2 \ \frac{d^2 f_2^{in}}{ds^2} + \frac{df_2^{in}}{ds} = 2s, f_2^{in}(0) = 0 \Rightarrow f_2^{in} = C - Ce^{-s} + s^2 - 2s$$

$$\varepsilon^n \ \frac{d^2 f_n^{in}}{ds^2} + \frac{df_n^{in}}{ds} = 0, f_n^{in}(0) = 0 \Rightarrow f_n^{in} = D_n - D_n e^{-s},$$

$n = 3, 4, \dots$

Prandtl's matching condition

To find the constant in the equations

$$f_0^{in} = A + (1 - A)e^{-s}$$

$$f_1^{in} = B - Be^{-s} + s$$

$$f_2^{in} = C - Ce^{-s} + s^2 - 2s$$

$$f_n^{in} = D_n - D_n e^{-s}, \quad n = 3, 4, \ldots$$

we can not use the boundary condition at $x = 1$. We need to match the inner and outer solutions. Let us do it for the leader terms $f_0^{in}(s)$ and $f_0^{out}(x)$.

Prandtl's matching condition

Let us chose two points $x = 5\varepsilon$ and $x = 6\varepsilon$ and equal $f_0^{out}(x) = x^2 + x + 2$ with $f_0^{in}(s) = A + (1 + A)e^{-s}$, $s = x/\varepsilon$
Then

$$A = \frac{2 + 5\varepsilon + 25\varepsilon^2 - e^{-5}}{1 - e^{-5}} \quad \text{for} \quad x = 5\varepsilon$$

$$A = \frac{2 + 6\varepsilon + 36\varepsilon^2 - e^{-6}}{1 - e^{-6}} \quad \text{for} \quad x = 6\varepsilon$$

Prandtl's matching condition

$$\lim_{x \to 0} f_0^{out}(x) = \lim_{s \to \infty} f_0^{in}(s)$$

Prandtl's matching condition

In our case it gives

$$\lim_{x\to 0} x^2 + x + 2 = \lim_{s\to\infty} A + (1+A)e^{-s} \quad \Rightarrow \quad A = 2.$$

So the leader terms are

Outer region $\quad f_0^{out} = x^2 + x + 2,\ $ for $\ x = O(1),$

Inner region $\quad f_0^{in} = 2 - e^{-x/\varepsilon},\ $ for $\ x = O(\varepsilon).$

Comparing with the exact solution

If $\ x = O(1)\ $ then $\ f_0^{ex} = x^2 + x + 2 + \dots,$

If $\ x = O(\varepsilon)\ $ then $\ f_0^{ex} = 2 - e^{-x/\varepsilon} + \dots,$

The composite expansion

We want to have

For $x = O(1)$ $f_0^{out} = f_0^{comp} +$ small terms,

For $x = O(\varepsilon)$ $f_0^{in} = f_0^{comp} +$ small terms,

If we choose $f_0^{comp} = f_0^{in} + f_0^{out} - f_0^{match}$, where f_0^{match} is given by the Prandtl's matching condition. Since

for $x = O(1)$ $f_0^{in} = f_0^{match} +$ small terms,

for $x = O(\varepsilon)$ $f_0^{out} = f_0^{match} +$ small terms.

For our example $f_0^{match} = 2$ and we get $f_0^{comp} = x^2 + x + 2 - e^{-x/\varepsilon}$.

How define where the boundary layer is located. Consider the example

$$\varepsilon\frac{d^2 f}{dx^2}+(x-2)\frac{df}{dx}+f=0,\ \varepsilon\ll 1,\ x\in(0,1),\ f(0)=3,\ f(1)=2$$

Let us suppose that boundary layer is at $x=0$. Then

$$(x-2)\frac{df_0^{out}}{dx}+f_0^{out}=0,\ f(1)=2 \quad\Rightarrow\quad f_0^{out}=\frac{2}{2-x}.$$

Use the stretched variables $s=x/\varepsilon$ to obtain new equation

Boundary layer location

$$\frac{d^2 f^{in}}{ds^2} + (\varepsilon s - 2)\frac{df^{in}}{ds} + \varepsilon f^{in} = 0.$$

The leading term f_0^{in} satisfies

$$\frac{d^2 f_0^{in}}{ds^2} - 2\frac{df_0^{in}}{ds} = 0, \qquad f_0^{in}(0) = 3$$

with solution

$$f_0^{in} = 3 - A + Ae^{2s}.$$

If $s \to \infty$, then $e^{2s} \to \infty$ very quickly. It happens because of the sign $-$ in the equation $\frac{d^2 f_0^{in}}{ds^2} - 2\frac{df_0^{in}}{ds} = 0$. So the choice of the location of the boundary layer at $x = 0$ was wrong.

Boundary layer location

Let us suppose that the boundary layer is located at $x = 1$. Then

$$(x-2)\frac{df_0^{out}}{dx} + f_0^{out} = 0,\ f(0) = 3 \quad \Rightarrow \quad f_0^{out} = \frac{6}{2-x}.$$

The appropriate stretched variable is $s = (1-x)/\varepsilon$. Then

$$\frac{df}{dx} = -\frac{1}{\varepsilon}\frac{f}{ds}, \qquad \frac{d^2 f}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2 f}{ds^2}.$$

The new equation is

$$\frac{d^2 f^{in}}{ds^2} + (\varepsilon s + 2)\frac{df^{in}}{ds} + \varepsilon f^{in} = 0.$$

Boundary layer location

The leading term f_0^{in} satisfies

$$\frac{d^2 f_0^{in}}{ds^2} + 2\frac{df_0^{in}}{ds} = 0, \qquad f_0^{in}(0) = 2 \;\; (x = 1 \mapsto s = 0)$$

with solution

$$f_0^{in} = 2 - A + Ae^{-s}.$$

Prandtl's matching condition applied to

$$f_0^{out} = \frac{6}{2-x} \;\; \text{and} \;\; f_0^{in} = 2 - A + Ae^{-s}$$

is

$$\lim_{x\to 1} \frac{6}{2-x} = \lim_{s\to\infty} 2 - A + Ae^{-s}$$

It gives $A = -4$ and

$$f_0^{comp} = \frac{6}{2-x} - 4e^{-(1-x)/\varepsilon}.$$

General linear equation

Some general consideration about boundary location of

$$\varepsilon \frac{d^2 f}{dx^2} + a(x)\frac{df}{dx} + b(x)f = c(x), \quad x_1 < x < x_2$$

I. If $a(x) > 0$ on the interval (x_1, x_2) then the boundary layer will occur at $x = x_1$. The stretching transformation is $s = (x - x_1)/\varepsilon$. The one term inner expansion will satisfy

$$\frac{d^2 f_0^{in}}{ds^2} + a(x_1)\frac{df_0^{in}}{ds} = 0.$$

The solution of the equation is

$$f_0^{in} = A + B\exp\left(-a(x_1)\frac{x - x_1}{\varepsilon}\right), \quad A + B = f(x = x_1).$$

General linear equation

$$\varepsilon \frac{d^2 f}{dx^2} + a(x)\frac{df}{dx} + b(x)f = c(x), \qquad x_1 < x < x_2$$

II. If $a(x) < 0$ on the interval (x_1, x_2) then the boundary layer will occur at $x = x_2$. The stretching transformation is $s = (x_2 - x)/\varepsilon$. The the inner expansion will involve term

$$\exp\left(a(x_2)\frac{x_2 - x}{\varepsilon}\right).$$

General linear equation

$$\varepsilon \frac{d^2 f}{dx^2} + a(x)\frac{df}{dx} + b(x) f = c(x), \quad x_1 < x < x_2$$

III. If $a(x)$ change the sign on the interval (x_1, x_2) then the boundary layer will occur at an interior point x_0 of (x_1, x_2) where $a(x_0) = 0$. Moreover the boundary layer can occur at both ends x_1 and x_2.

The end