Multiple scale technique

Time scales

Lesson 12

The basic idea of the multiple scale technique is to preserve the combinations εt, $\varepsilon^2 t$, $\varepsilon^3 t$ as variables on which this function depends. We move from the one variable, that unbounded to the two or three, which will be bounded and by this do not produce secular terms. We denote

$$T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t,$$

and consider the function

$$f = \frac{1}{1-\varepsilon}\exp(t + \varepsilon t + \varepsilon^2 t + \varepsilon^3 t + \varepsilon^4 t).$$

We use the one scale $T_0 = t$. Then

$$f = \frac{1}{1-\varepsilon}\exp T_0 \exp(\varepsilon T_0 + \varepsilon^2 T_0 + \dots)$$

$$\sim \Big(1 + \varepsilon + \frac{\varepsilon^2}{2} + \dots\Big)\exp T_0 \cdot \Big((1 + \varepsilon T_0 + \varepsilon^2 T_0 + \dots)$$

$$+ \frac{1}{2}(\varepsilon T_0 + \varepsilon^2 T_0 + \dots)^2 + \dots\Big)$$

$$\sim \exp T_0 + \varepsilon(1 + T_0)\exp T_0 + \varepsilon^2(\frac{1}{2} + T_0 + \frac{3}{2}T_0^2)\exp T_0 + \dots.$$

The region of nonuniformity is $T_0 = O(1/\varepsilon)$ or $t = O(1/\varepsilon)$. Let us try the two scale expansion, when $T_1 = \varepsilon T_0$ and $T_1 = O(\varepsilon)$.

Then

$$f = \frac{1}{1-\varepsilon}\exp(T_0+T_1)\exp(\varepsilon^2 T_0+\varepsilon^3 T_0+\dots)$$

$$\sim \Big(1+\varepsilon+\frac{\varepsilon^2}{2}+\dots\Big)\exp(T_0+T_1)\cdot\Big((1+\varepsilon^2 T_0+\dots)$$

$$+\frac{1}{2}(\varepsilon^2 T_0+\dots)^2+\dots\Big)$$

$$\sim \exp(T_0+T_1)+\varepsilon\exp(T_0+T_1)+\varepsilon^2(\frac{1}{2}+T_0)\exp(T_0+T_1)+$$

If $T_0 = O(1/\varepsilon)$ then the expansion is uniform and

$$f = \exp(T_0+T_1)\cdot\big(1+O(\varepsilon,\varepsilon^2 T_0)\big) \text{ uniform } t = O(1/\varepsilon).$$

We can use three scale expansion $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$ and we get

$$f = \exp(T_0 + T_1 + T_2) \cdot \big(1 + \varepsilon + O(\varepsilon^2, \varepsilon^3 T_0)\big),$$

which is uniform for $t = O(1/\varepsilon^2)$. We have a choice

1. increase the number of scales variables
2. use the expansion with fewer number of terms.

Let us apply the multiple scale technique to the example of the previous lecture.

Model example

$$\frac{d^2v}{dt^2} + v = -\varepsilon\frac{dv}{dt}, \quad v(0) = 1, \quad \frac{dv}{dt}(0) = 0.$$

We shall restrict the consideration to two scales $T_0 = t$ and $T_1 = \varepsilon t$. The derivatives

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1},$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2} \quad \Rightarrow$$

$$\frac{\partial^2 v}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 v}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2 v}{\partial T_1^2} + v = -\varepsilon\left(\frac{\partial v}{\partial T_0} + \varepsilon\frac{\partial v}{\partial T_1}\right).$$

Model example

$$\frac{\partial^2 v}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 v}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 v}{\partial T_1^2} + v = -\varepsilon\Big(\frac{\partial v}{\partial T_0} + \varepsilon \frac{\partial v}{\partial T_1}\Big).$$

Use the expansion $v(T_0, T_1, \varepsilon) \sim v_0(T_0, T_1) + \varepsilon v_1(T_0, T_1)$.

$$\varepsilon^0: \ \frac{\partial^2 v_0}{\partial T_0^2} + v_0 = 0, \quad v_0(0,0) = 1, \quad \frac{\partial v_0}{\partial T_0}(0,0) = 0$$

$$\varepsilon^1: \ \frac{\partial^2 v_1}{\partial T_0^2} + v_1 = -2\frac{\partial^2 v_0}{\partial T_0 \partial T_1} - \frac{\partial v_0}{\partial T_0}, \quad v_1(0,0) = 0,$$

$$\frac{\partial v_1}{\partial T_0}(0,0) = \frac{\partial v_1}{\partial T_0}(0,0) = 0.$$

Model example

The solution of the first equation has the same form as the homogeneous solution of the second equation. We write the solution of the first equation in the complex form

$$v_0 = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0}.$$

The right hand side of the second equation is

$$-2i\left(\frac{dA}{dT_1}e^{iT_0} - \frac{dA^*}{dT_1}e^{-iT_0}\right) - i\left(Ae^{iT_0} - A^*e^{-iT_0}\right)$$

$$= -i\left(2\frac{dA}{dT_1} + A\right)e^{iT_0} + i\left(2\frac{dA^*}{dT_1} + A^*\right)\, e^{-iT_0}.$$

To avoid secular terms we required

$$2\frac{dA}{dT_1} + A = 0, \qquad 2\frac{dA^*}{dT_1} + A^* = 0.$$

We write A in a polar form $A(T_1) = R(T_1)e^{i\theta(T_1)}$. Then

$$\frac{dA}{dT_1} = \left(\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}\right)e^{i\theta} \Rightarrow \left(2\frac{dR}{dT_1} + 2iR\frac{d\theta}{dT_1} + R\right)e^{i\theta} = 0$$

$$2\frac{dR}{dT_1} + R = 0 \qquad \text{and} \qquad 2R\frac{d\theta}{dT_1} = 0$$

$$2R\frac{d\theta}{dT_1} = 0 \quad \Rightarrow \quad \theta = \theta_0,$$

$$2\frac{dR}{dT_1} + R = 0 \quad \Rightarrow \quad R = R_0 e^{-T_1/2},\ \ R_0 > 0.$$

We get

$$A(T_1) = R_0 e^{-T_1/2} e^{i\theta_0}.$$

Funding $A(T_1)$ we avoid the secular terms. Substituting $A(T_1)$ into the formula for

$$v_0 = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0}$$

we get

Model example

$$v_0 = R_0 e^{-T_1/2}\left(e^{i(T_0+\theta_0)}+e^{-i(T_0+\theta_0)}\right) = 2R_0 e^{-T_1/2}\cos(T_0+\theta_0).$$

Now we exploit the initial conditions $v_0(0,0) = 1$ and $\frac{\partial v_0}{\partial T_0}(0,0) = 0$, getting

$$2R_0\cos\theta_0 = 1,\ -2R_0\sin\theta_0 = 0 \ \Rightarrow\ \theta_0 = 0, \quad R_0 = 1/2.$$

The one term expansion is

$$v_0 = e^{-T_1/2}\cos T_0 + O(\varepsilon) \quad \text{or} \quad v_0 = e^{-\varepsilon t/2}\cos t + O(\varepsilon).$$

We do not use v_1 to keep the uniformity.

$$\frac{d^2u}{dt^2} + u = \varepsilon(1-u^2)\frac{du}{dt}, \quad u(0) = a, \quad \frac{du}{dt}(0) = 0.$$

We use two scales $T_0 = t$ and $T_1 = \varepsilon t$.

$$\frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 u}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2 u}{\partial T_1^2} + u = \varepsilon(1-u^2)\Big(\frac{\partial u}{\partial T_0} + \varepsilon\frac{\partial u}{\partial T_1}\Big).$$

Substituting $u(T_0, T_1, \varepsilon) \sim u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1)$.

$$\varepsilon^0: \ \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0,$$

$$\varepsilon^1: \ \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + (1-u^2)\frac{\partial u_0}{\partial T_0},$$

$$u_0 = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0}.$$

The right hand side of the second equation is

$$- 2i\Big(\frac{dA}{dT_1}e^{iT_0} - \frac{dA^*}{dT_1}e^{-iT_0}\Big) + i\Big(1 - A^2e^{2iT_0} - 2AA^*$$
$$+ A^{*2}e^{-2iT_0}\Big)\Big(Ae^{iT_0} - A^*e^{-iT_0}\Big)$$
$$= \Big(- 2i\frac{dA}{dT_1} + iA - iAA^*\Big)e^{iT_0}$$
$$+ \Big(- 2i\frac{dA}{dT_1} + iA - iAA^*\Big)^*e^{-iT_0} + \text{(nonsecular terms)}.$$

Van der Pol oscillator

Using the polar form $A = Re^{\theta}$, we write

$$-2\left(\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}\right) + R - R^3 = 0 \quad \Rightarrow$$

$$-2\frac{dR}{dT_1} + R - R^3 = 0, \quad R\frac{d\theta}{dT_1} = 0 \quad \Rightarrow$$

The solutions are

$$\theta = \theta_0, \qquad \frac{dR}{R(R^2-1)} = -\frac{dT_1}{2} \quad \Rightarrow$$

- $\int \left(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \right) dR = -\int \frac{dT_1}{2}$,

Van der Pol oscillator

- $\int \left(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \right) dR = -\int \frac{dT_1}{2}$,
- $-\ln|R| + \frac{1}{2}\ln|R-1| + \frac{1}{2}\ln|R+1| = -\frac{T_1}{2} + C$,

Van der Pol oscillator

- $\int \left(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \right) dR = -\int \frac{dT_1}{2},$
- $-\ln |R| + \frac{1}{2} \ln |R-1| + \frac{1}{2} \ln |R+1| = -\frac{T_1}{2} + C,$
- $\ln \sqrt{\left| \frac{R^2-1}{R^2} \right|} = -\frac{T_1}{2} + C,$

Van der Pol oscillator

- $\int \Big(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \Big) dR = -\int \frac{dT_1}{2}$,
- $-\ln|R| + \frac{1}{2}\ln|R-1| + \frac{1}{2}\ln|R+1| = -\frac{T_1}{2} + C$,
- $\ln\sqrt{|\frac{R^2-1}{R^2}|} = -\frac{T_1}{2} + C$,
- $\frac{R^2-1}{R^2} = Ke^{-T_1} \;\Rightarrow\; R^2 = \frac{1}{1-Ke^{-T_1}}$,

Van der Pol oscillator

- $\int \left(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \right) dR = -\int \frac{dT_1}{2}$,
- $-\ln|R| + \frac{1}{2}\ln|R-1| + \frac{1}{2}\ln|R+1| = -\frac{T_1}{2} + C$,
- $\ln\sqrt{|\frac{R^2-1}{R^2}|} = -\frac{T_1}{2} + C$,
- $\frac{R^2-1}{R^2} = Ke^{-T_1} \;\Rightarrow\; R^2 = \frac{1}{1-Ke^{-T_1}}$,
- $u_0 = Re^{i(T_0+\theta_0)} + Re^{-i(T_0+\theta_0)} = 2R\cos(T_0+\theta_0)$.

Van der Pol oscillator

- $\int \Big(-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)} \Big) dR = -\int \frac{dT_1}{2}$,
- $-\ln|R| + \frac{1}{2}\ln|R-1| + \frac{1}{2}\ln|R+1| = -\frac{T_1}{2} + C$,
- $\ln \sqrt{|\frac{R^2-1}{R^2}|} = -\frac{T_1}{2} + C$,
- $\frac{R^2-1}{R^2} = Ke^{-T_1} \;\Rightarrow\; R^2 = \frac{1}{1-Ke^{-T_1}}$,
- $u_0 = Re^{i(T_0+\theta_0)} + Re^{-i(T_0+\theta_0)} = 2R\cos(T_0+\theta_0)$.
- The initial conditions $u_0(0,0) = a$, $\frac{\partial u_0}{\partial T_0}(0,0) = 0$ gives

Van der Pol oscillator

$$2R(0)\cos\theta_0 = a,\ -2R(0)\sin\theta_0 = 0 \quad\Rightarrow\quad \theta_0 = 0,\ R(0) = a/2.$$

$$R^2 = \frac{1}{1-Ke^{-T_1}} \ \Rightarrow\ a^2/4 = \frac{1}{1-Ke^{-T_1}} \ \Rightarrow\ K = 1 - 4/a^2,$$

$$u_0 = \frac{2\cos T_0}{\sqrt{1+\left(\frac{4}{a^2}-1\right)e^{-T_1}}}, \quad u_1 = O(1), \quad \forall t.$$

$$u = \frac{2\cos t}{\sqrt{1+\left(\frac{4}{a^2}-1\right)e^{-\varepsilon t}}} + O(\varepsilon).$$

The expansion

$$u = \frac{2\cos t}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right)e^{-\varepsilon t}}} + O(\varepsilon)$$

tends to the limit cycle $u = 2\cos t + O(\varepsilon)$. If $a = 2$ then the solution is periodic $u = 2\cos t$.

Van der Pol oscillator

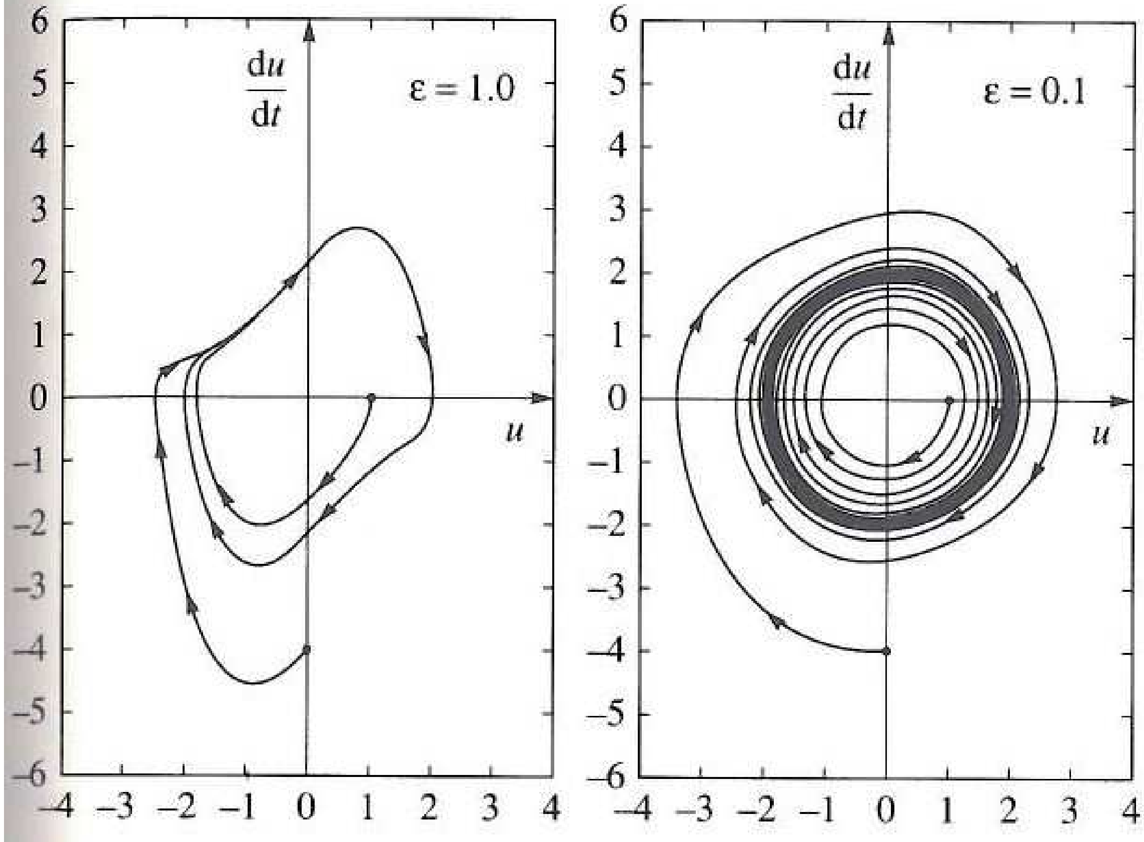

Fig. 4.4 Solutions of the van der Pol equation

Van der Pol oscillator

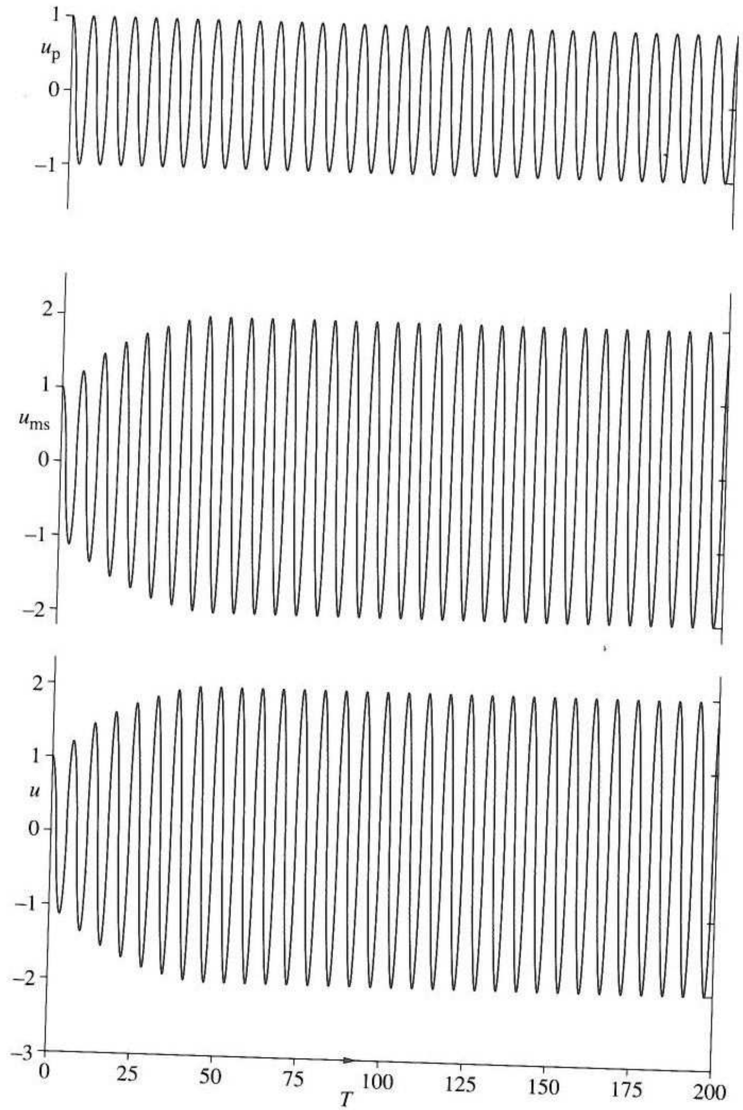

Fig. 4.5 Comparison of the numerical solution, u, the straightforward expansion, u_p, and the multiple scale expansion, u_{ms}, of the van der Pol equation with $\varepsilon = 0.1$.

Duffing's equation

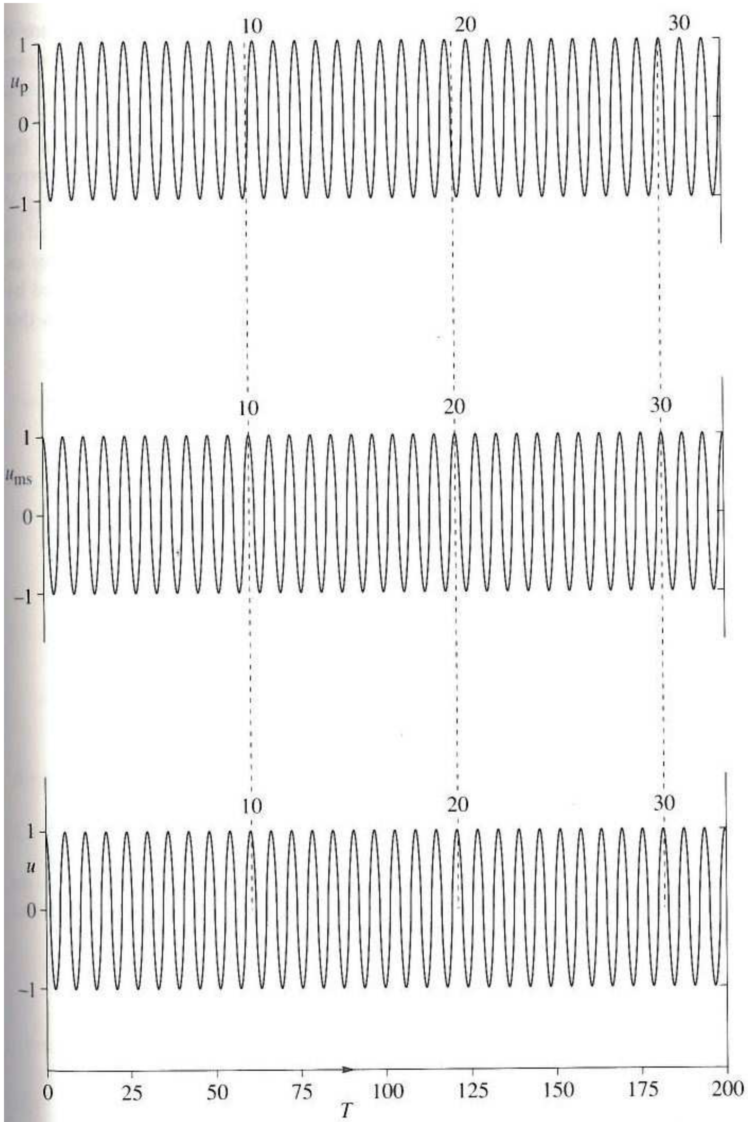

Fig. 4.6 Comparison of the numerical solution, u, the straightforward expansion, u_p, and the multiple scale expansion, u_{ms}, of Duffing's equation with $\varepsilon = 0.1$.

The end