Multiple scale technique
Second order systems

Lesson 11

Weakly nonlinear oscillator

$$\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon F(u, \frac{du}{dt}).$$

it is a general form of Duffing's equation

$$\frac{d^2u}{dt^2} + u \mp \varepsilon u^3 = 0 \text{ with } F = \pm u^3$$

or van der Pol's oscillator

$$\frac{d^2u}{dt^2} + u - \varepsilon(1-u^2)\frac{du}{dt} = 0 \text{ with } F = (1-u^2)\frac{du}{dt}.$$

For the first equation the method of renormalization overcomes the problem of nonuniformity, but for the second equation it fails.

Type of solutions of the second order ODE.

The multiple scale technique permits successfully treat all types of weakly nonlinear oscillators. But before introducing the method we review the types of the the second order ODE.
We start from the unperturbed equation

$$\frac{d^2u}{dt^2} + \omega_0^2 u = 0.$$

Its solution is

$$u = A\cos\omega_0 t + B\sin\omega_0 t \text{ with } A = u(0),\ B = \frac{1}{\omega_0}\frac{du}{dt}(0).$$

Let us introduce the coordinate system with $x = u$ and $y = \frac{du}{dt}$. Then

$$x = u = A\cos\omega_0 t + B\sin\omega_0 t$$

and

$$y = \frac{du}{dt} = -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t.$$

Eliminating t yields the solution curve on (x, y) plane

$$x^2 + \frac{y^2}{\omega_0^2} = A^2 + B^2.$$

The phase plane

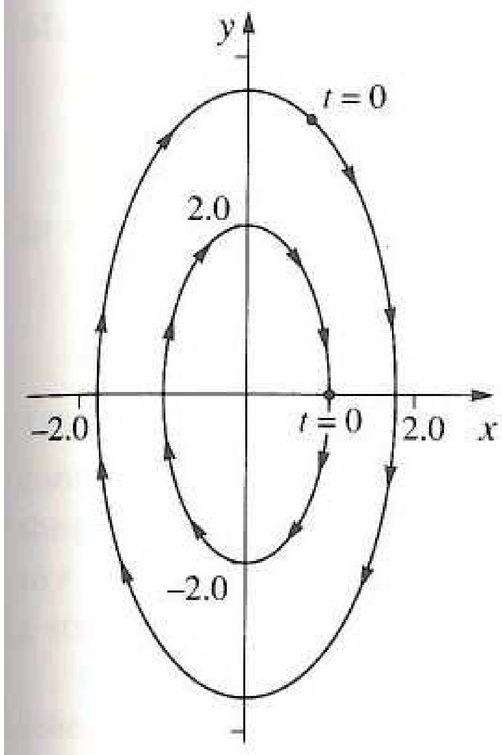

Fig. 4.1 Phase plane solution curves of equation 4.1.2

The curves $x^2 + \frac{y^2}{\omega_0} = A^2 + B^2$ are ellipses with semi-axes $\sqrt{A^2 + B^2}$ and $\omega_0\sqrt{A^2 + B^2}$. The directions of the trajectories is determined by the sign of y.

The differential equation $\frac{d^2u}{dt^2} + \frac{du}{dt} + 2u = 0$ has the general solution

$$u = e^{-\frac{t}{2}}\left(A\cos\frac{\sqrt{7}}{2}t + B\sin\frac{\sqrt{7}}{2}t\right)$$

where

$$A = u(0) \text{ and } -\frac{A}{2} + \frac{\sqrt{7}B}{2} = \frac{du}{dt}(0)$$

The phase plane

as we see the solutions are the spiral into the origin

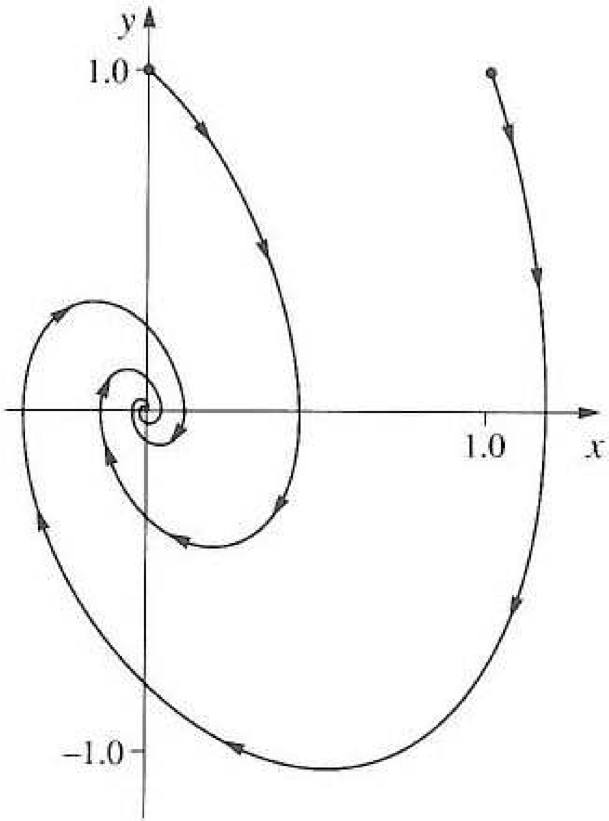

Fig. 4.2 Phase plane solution curves of equation 4.1.3

Linear autonomous equations

There are three types of behavior of solutions of autonomous linear differential equations:

1. periodic orbits,
2. decay towards the origin,
3. diverge to the infinity.

Nonlinear autonomous equations

The nonlinear autonomous differential equations has one more special type of solutions *limit cycle*. Occurrence of this type depends both on the form of the equation and on the initial conditions. As an example consider the equation

$$\frac{d^2u}{dt^2} = -u + \left(1 - u^2 - (\frac{du}{dt})^2\right)\frac{du}{dt}.$$

Nonlinear autonomous equations

Let us write the equation $\frac{d^2u}{dt^2} = -u + \left(1 - u^2 - \left(\frac{du}{dt}\right)^2\right)\frac{du}{dt}$ using the phase plane variables

$$x = u, \quad y = \frac{dx}{dt} = \frac{du}{dt}.$$

Then the equation is equivalent to the system

$$\frac{dx}{dt} = y \qquad\qquad | \cdot x$$
$$\frac{dy}{dt} = -x + (1 - x^2 - y^2)y \qquad\qquad | \cdot y$$

Nonlinear autonomous equations

$$\frac{d}{dt}(x^2 + y^2) = 2(1 - x^2 - y^2)y^2.$$

Using the polar coordinates

$$x = r\cos\theta$$
$$r = \sin\theta$$

we get the

$$\frac{d}{dt}(r^2) = 2(1 - r^2)r^2\sin^2\theta.$$

Nonlinear autonomous equations

There is three possibility for the equation
$\frac{d}{dt}(r^2) = 2(1 - r^2)r^2 \sin^2\theta$

1. If $r = 1$ then $\frac{d}{dt}(r^2) = 0$ and $r = const$ give the periodic solution.
2. If $r > 1$ then $\frac{d}{dt}(r^2) < 0$ and the solution decrease to the periodic solution.
3. If $r < 1$ then $\frac{d}{dt}(r^2) > 0$ and the solution increase to the periodic solution.

This situation called the stable limit cycle.

Limit cycle

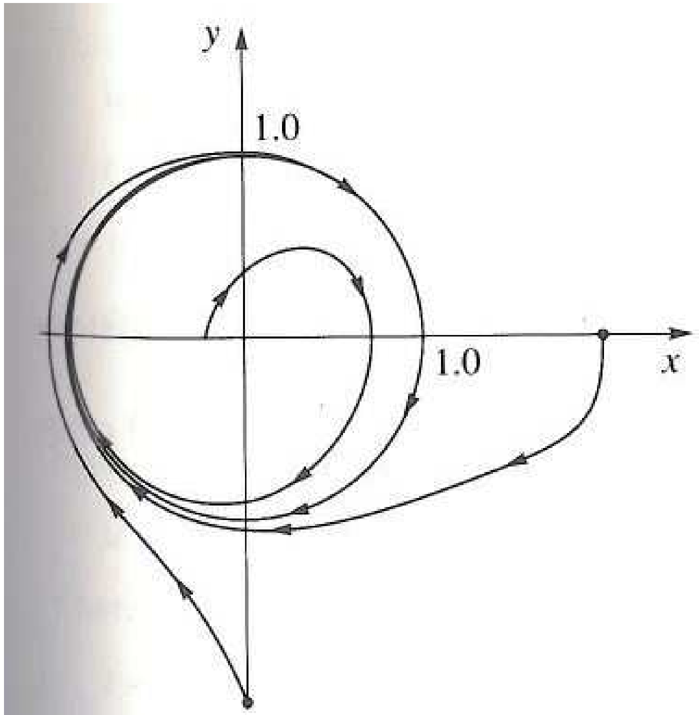

Fig. 4.3 Trajectories approaching the limit cycle of equation 4.1.4

Types of solutions for nonlinear ODE

The solutions of nonlinear autonomous second order differential equations belong to the following four classes

1. periodic solutions,
2. limit cycles,
3. solutions tends towards a fixed value
4. solutions tends to infinity.

The renormalization is success only for the first type of solutions.

Limitation of renormalization

Let us compare two problems

$$\frac{d^2u}{dt^2} + u = \varepsilon u, \quad u(0) = 1, \quad \frac{du}{dt}(0) = 0$$
$$\frac{d^2v}{dt^2} + v = -\varepsilon \frac{dv}{dt}, \quad v(0) = 1, \quad \frac{dv}{dt}(0) = 0$$

We look for the two term expansions

$$u(t, \varepsilon) = u_0(t) + \varepsilon u_1(t)$$

and

$$v(t, \varepsilon) = v_0(t) + \varepsilon v_1(t).$$

Then $u_0(t) = v_0(t) = \cos t$.

Limitation of renormalization

The second coefficients are following

$$\frac{d^2u_1}{dt^2} + u_1 = \cos t, \quad u_1(0) = \frac{du_1}{dt}(0) = 0 \quad \Rightarrow$$

$$u_1 = \frac{t}{2}\sin t,$$

$$\frac{d^2v_1}{dt^2} + v_1 = -\sin t, \quad v_1(0) = \frac{dv_1}{dt}(0) = 0 \quad \Rightarrow$$

$$v_1 = \frac{1}{2}(-t\cos t + \sin t).$$

Limitation of renormalization

Introducing the strained coordinates $t = s + \varepsilon f_1$, we get for the first equation

$$\cos t + \frac{1}{2}\varepsilon t \sin t = \cos(s + \varepsilon f_1) + \frac{1}{2}\varepsilon s \sin s$$
$$= \cos s - \varepsilon f_1 \sin s + \frac{1}{2}\varepsilon s \sin s$$

Choosing $f_1 = \frac{s}{2}$ we have $u = \cos s + O(\varepsilon)$ where

$$t = s + \varepsilon \frac{s}{2} + O(\varepsilon^2)$$

Limitation of renormalization

Expressing s

$$s = t\left(1 + \frac{\varepsilon}{2} + O(\varepsilon^2)\right)^{-1} = t\left(1 - \frac{\varepsilon}{2} + O(\varepsilon^2)\right).$$

Thus

$$u = \cos t\left(1 - \frac{\varepsilon}{2} + \dots\right).$$

The exact solution is

$$u = \cos t\sqrt{1-\varepsilon} = \cos t\left(1 - \frac{\varepsilon}{2} + \dots\right).$$

The renormalization provides a means of evaluating the frequency change associated with perturbed system whose solution are periodic.

Limitation of renormalization

Introducing the strained coordinates $t = s + \varepsilon f_1$ for the second equation

$$\cos t - \frac{1}{2}\varepsilon(t\cos t - \sin t) = \cos(s + \varepsilon f_1) - \frac{1}{2}\varepsilon(s\cos s - \sin s)$$
$$= \cos s - \varepsilon f_1 \sin s - \frac{1}{2}\varepsilon s\cos s + \frac{1}{2}\varepsilon\sin s$$

$f_1 = -\frac{s}{2}\cot s$ is not acceptable since it ia singular for $s = \pi, 2\pi, 3\pi, \ldots$. Trying to make the expansion of solution uniform the strained expansion becomes nonuniform. Let us compare with the exact solution.

Time changing amplitude

$$u = e^{-\frac{\varepsilon t}{2}} \left(\cos \sqrt{1 - \frac{\varepsilon^2}{4}} t + \frac{\varepsilon}{2\sqrt{1 - \frac{\varepsilon^2}{4}}} \sin \sqrt{1 - \frac{\varepsilon^2}{4}} t \right).$$

It is not a periodic solution since the frequency of oscillation and the amplitude are changed.

The frequency of oscillation changed from 1 to $\sqrt{1 - \frac{\varepsilon^2}{4}}$

The amplitude $e^{-\frac{\varepsilon t}{2}}$ decays.

Let us see why the renormalization failed in general case.

Limitation of renormalization

$$\frac{d^2u}{dt^2} + u = \varepsilon F(u, \frac{du}{dt}), \quad u(0) = a, \quad \frac{du}{dt}(0) = 0$$

Starting from two term expansion $u_0 + \varepsilon u_1$ we get $u_0 = a \cos t$ and

$$\frac{d^2u_1}{dt^2} + u_1 = \varepsilon F(a \cos t, -a \sin t).$$

Since F is a periodic function, it can be decomposed into the Fourier series

$$F(a \cos t, -a \sin t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big).$$

Limitation of renormalization

The homogeneous solution is $A\cos t + B\sin t$, so the term $a_1\cos t + b_1\sin t$ will generate a particular solution

$$\frac{t}{2}(a_1\sin t - b_1\cos t).$$

The two term expansion is

$$a\cos t + \varepsilon\Big(\frac{t}{2}(a_1\sin t - b_1\cos t) + \text{ nonsecular terms}\Big)$$

Using the renormalization $t = s + \varepsilon f_1$ we get

$$a\cos s - \varepsilon a f_1 \sin s + \varepsilon\frac{s}{2}(a_1\sin s - b_1\cos s) + \text{ nonsecular terms.}$$

Limitation of renormalization

We require $f_1 = \frac{a_1 s}{2a} - b_1 s \cot \frac{s}{2a}$. We see that iff $b_1 = 0$ we avoid the problems with $\cot \frac{s}{2a}$. In this case the expansion is

$$u = a \cos s + O(\varepsilon), \quad t = s + \varepsilon \frac{a_1 s}{2a} + O(\varepsilon^2) \quad \Rightarrow$$

$$u = a \cos t \Big(1 - \varepsilon \frac{a_1}{2a} + \dots \Big).$$

The expansion can generate only periodic solutions.

For Duffing's equation we had $F = -u^3$, so

$$F(a\cos t, -a\sin t) = -a^3\cos^3 t = \frac{a^3}{4}(3\cos t + \cos 3t)$$

and renormalization gave the positive result.
For van der Pol's oscillator $F = (1-u^2)\frac{du}{dt}$

$$F = (1 - a^2\cos^2 t)(-a\sin t) = (-a + \frac{a^3}{4})\sin t + \frac{a^3}{4}\sin 3t.$$

If $a \neq 2$ the renormalization fails. The case $a = 2$ corresponds to the limit cycle and the normalization is successful.

The end