Good and bad sides of renormalization

Lesson 10

Application of renormalization to nonlinear oscillator

Consider the equation

$$\frac{d^2u}{dt^2} + u = \varepsilon u\Big(1 - (\frac{du}{dt})^2\Big), \quad u(0) = a, \frac{du}{dt}(0) = 0.$$

Let us apply the renormalization technique. We start from the standard expansion $u(t,\varepsilon) \sim u_0(t) + \varepsilon u_1(t)$

$$\varepsilon^0: \; \frac{d^2u_0}{dt^2} + u_0 = 0, \; u_0(0) = a, \; \frac{du_0}{dt}(0) = 0 \;\; \Rightarrow \;\; u_0 = a\cos t,$$

$$\varepsilon^1: \; \frac{d^2u_1}{dt^2} + u_1 = u_0\Big(1 - (\frac{du_0}{dt})^2\Big), \; u_1(0) = \frac{du_1}{dt}(0) = 0 \;\; \Rightarrow$$

Nonlinear oscillator

$$\frac{d^2u_1}{dt^2} + u_1 = a\cos t - a^3 \cos t \sin^2 t.$$

We calculate

$$\cos t \sin^2 t = \frac{1}{2\cdot(2i)^2}(e^{it}+e^{-it})(e^{it}-e^{-it})^2$$

$$= -\frac{1}{8}(e^{3it} - 2e^{it} + e^{-3it} - 2e^{-it}) = -\frac{1}{4}\cos 3t + \frac{1}{2}\cos t.$$

Thus $\frac{d^2u_1}{dt^2} + u_1 = -\frac{a^3}{4}\cos 3t + \left(a + \frac{a^3}{2}\right)\cos t$

$$\frac{d^2u_1}{dt^2} + u_1 = -\frac{a^3}{4}\cos 3t + \left(a + \frac{a^3}{2}\right)\cos t$$

$A\cos t + B\sin t$ homogeneous solution

$\alpha\cos 3t + \beta t\sin t$ particular solution.

Applying the initial conditions $u(0) = \frac{du}{dt}(0) = 0$ we deduce $\alpha = \frac{a^3}{32}$, $\beta = \frac{a}{2} + \frac{a^3}{4}$, $A = -\frac{a^3}{32}$, $B = 0$.

$$u = a\cos t + \varepsilon\left(\frac{a^3}{32}(\cos 3t - \cos t) + \left(\frac{a}{2} + \frac{a^3}{4}\right)t\sin t\right)$$

The region of nonuniformity is $t = O(1/\varepsilon)$.

$$u = a\cos t + \varepsilon\Big(\frac{a^3}{32}(\cos 3t - \cos t) + (\frac{a}{2} + \frac{a^3}{4})t\sin t\Big)$$

We introduce the strained coordinates

$t \sim s + \varepsilon f_1(s) + \ldots$ then $u \sim a\cos s$

$$-a\varepsilon f_1 \sin s + \ldots + \varepsilon\Big(\frac{a^3}{32}(\cos 3s - \cos s) + (\frac{a}{2} + \frac{a^3}{4})s\sin s + \ldots\Big)$$

The secular term removed if $f_1 = s(\frac{1}{2} + \frac{a^2}{4})$

The one-term uniformly valid expansion is $u = a\cos s + O(\varepsilon)$ where

$$t = s + \varepsilon s\left(\frac{1}{2} + \frac{a^2}{4}\right) + O(\varepsilon)$$

or

$$s = t\left(1 - \varepsilon\left(\frac{1}{2} + \frac{a^2}{4}\right) + O(\varepsilon)\right)$$

Finally

$$u = a\cos t\left(1 - \varepsilon\left(\frac{1}{2} + \frac{a^2}{4}\right) + \dots\right) + O(\varepsilon) \quad \varepsilon \to 0.$$

Failure of renormalization

Consider the van der Pol oscillator

$$\frac{d^2u}{dt^2} + u = \varepsilon(1-u^2)\frac{du}{dt}, \qquad u(0) = 1, \frac{du}{dt}(0) = 0.$$

The straightforward two term expansion is

$$u = \cos t + \varepsilon\left(\frac{3}{8}t\cos t - \frac{9}{32}\sin t - \frac{1}{32}\sin 3t\right).$$

The region of nonuniformity is $t = O(1/\varepsilon)$.

Failure of renormalization

We introduce the strained coordinates $t \sim s + \varepsilon f_1(s) + \ldots$ then

$$u \sim \cos s - \varepsilon f_1 \sin s + \varepsilon\left(\frac{3}{8}s\cos s - \frac{9}{32}\sin s - \frac{1}{32}\sin 3s\right)$$

The secular term is

$$-f_1 \sin s + \frac{3}{8}s\cos s = 0 \quad \Rightarrow \quad f_1 = \frac{3}{8}s\cos s \;\text{ then}$$

$$u = \cos s + O(\varepsilon) \;\text{ where }\; t = s + \frac{3}{8}\varepsilon s\cot s + O(\varepsilon^2).$$

The cotangent function is singular when $s = 0, \pi, 2\pi, \ldots$

The end