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Wave breaking in the KdV equation on a flow with constant vorticity

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A R T I C L E I N F O

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A B S T R A C T

The paper describes the critical breaking waveheight for long surface water waves on a flow with constant vorticity in the KdV approximation. Given a background linear shear flow, a KdV equation can be found with coefficients depending on the strength of the shear flow. The derivation also shows that the velocity field under the wave can be constructed approximately from the free surface excursion.

A convective breaking criterion is put forward and used to detect incipient wave breaking in periodic traveling waves and solitary waves. It is shown that for both the solitary wave and the cnoidal waves, there are limiting waveheights where the horizontal component of the particle velocity equals the phase velocity of the wave. It is found that the strength of the vorticity has a considerable influence on the critical waveheight.

1. Introduction

Boussinesq-type models are widely used in coastal hydrodynamics for predicting the transformation of free-surface waves from deeper to shallower water. While there is a great variety of models including dispersion-optimized models and higher-order models which may be able to capture large-amplitude effects, many models in use today do not take into account vorticity. Of course, in nature, most coastal flows do feature vorticity, and there have been efforts to incorporate various forms of vorticity into the mathematical description of the flow, such as for example in [1,2].

One of the first works to include the effects of shear flows into shallow-water theory was [3], which examined the range of characteristic speeds for general shear profiles. In [4] the authors suggested the configuration of a free surface over a constant stream with prescribed vorticity as an approximate model for a larger variety of flows with background vorticity. While linear shear flows may not regularly occur in nature, it was argued in [4] that a linear background stream can be used as a first approximation to more general shear profiles in the case when the wavelength of the waves is on a different scale than the variation of the shear profile. In particular for long waves such as considered in the present paper, there is a scale separation between the wavelength and the typical variation of a shear flow, so that the mean vorticity has much greater influence on the wave dynamics than the specific distribution of vorticity.

Following the lead of [4], constant background shear has been used by many authors in order to incorporate vorticity into their models (see for example [5–9] and the references therein). Other examples of cases which have proved to be mathematically tractable include compactly supported vorticity, such as point vortices or vortex patches [10,11], and the creation of vorticity through interaction with bathymetry [12] or through singular flow such as hydraulic jumps [13].

The main purpose of the present work is to investigate the effect of background vorticity on wave breaking at the free surface. For the sake of simplicity, our study will be conducted in the context of the KdV equation to which the original Boussinesq system reduces in the case of waves traveling in a single specific direction [14]. Wave breaking is understood in terms of stagnation at the free surface, when the horizontal particle velocity at the wavecrest reaches the same value as the phase velocity. In order to test such a kinematic breaking criterion in the context of the KdV equation, it is necessary to have an estimate for the horizontal component of the velocity field in the KdV approximation.

Derivations of the KdV equation in the presence of background velocity exist [15], but here the focus is on the reconstruction of the velocity field, so we provide an independent derivation which also yields an expression for the horizontal velocity in terms of the principal unknown variable $\eta(x,t)$ which describes the deflection of the free surface at a time t and a spatial position x. Once the horizontal velocity is known, it can be used to evaluate the kinematic wave-breaking criterion which predicts wave breaking when the horizontal component of the particle velocity at the wave crest exceeds the crest speed [16].

Wave breaking is investigated for both solitary waves and periodic cnoidal waves, and it is found that if the shear is favorable, i.e. in the direction of wave propagation, wave breaking is inhibited. On the other hand, a shear flow in the direction opposite of

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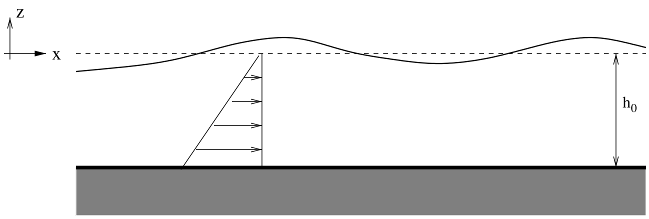

Fig. 1. The background uniform shear flow $U = z\Gamma$. In the figure Γ is negative. In the KdV equation, the waves which are superposed onto this background current propagate to the right.

the wave velocity has the effect of facilitating the breaking of the wave. This finding is in agreement with previous studies of wave breaking through stagnation points using the full Euler equations. In particular, the numerical approach used in [17] also shows the same qualitative influence of the background vorticity on the onset of wave breaking.

The accuracy of the kinematic breaking criterion has been discussed a great deal in the literature. In some investigations, it was observed that the kinematic criterion can be useful in some practical situations, while there is also evidence that it may not always be a reliable indicator of the onset of breaking. However, some studies, such as [18] which focus on breaking in shallow water conclude that the convective criterion does a fair job of predicting wave breaking even for three-dimensional waves. Apparently, if the kinematic criterion fails to predict the onset of wave breaking accurately, the problem often lies with the difficulty of obtaining accurate measurements for the phase velocity of the wave.

Wave steepness has also been used as an indicator for the occurrence of wave breaking, primarily for waves in intermediate and deep water, but recent findings cast doubt on the use of this measure to predict wave breaking [19]. In the same paper, the kinematic criterion is also called into question, and a criterion based on the ratio of energy density and energy flux is proposed. This idea actually leads to a tightened kinematic criterion, such as found in [20] and used also in [21].

2. Formulation of the mathematical problem

Let us briefly recall the governing equations for two dimensional water waves with constant vorticity. The starting point for the mathematical formulation are the Euler equations in two dimensions with no-penetration conditions at the bed and kinematic and dynamic boundary conditions at the free surface. Let the spatial coordinates be (x, z) and the x-axis be oriented in the horizontal direction and aligned with the undisturbed free surface (see Fig. 1). Let $\eta(x, t)$ denote the surface elevation, and assume that the motion is uniform in the direction perpendicular to the xz-plane (long-crested waves). As usual, the gravitational acceleration g is directed in the negative z-direction.

A shear flow with constant vorticity is prescribed, and the velocity field is given by $\mathbf{u} = (U, W) = (u + z\Gamma, w) = (\frac{\partial\phi}{\partial x} + z\Gamma, \frac{\partial\phi}{\partial z})$, where $\phi(x, z, t)$ is the velocity potential of the irrotational disturbance, and Γ is fixed. The potential part of the velocity field satisfies the problem

$$\Delta\phi = 0, \quad -h_0 < z < \eta,$$
$$\phi_z = 0, \quad z = -h_0.$$

In addition, assuming atmospheric pressure is normalized to zero, we have the dynamic and kinematic conditions

$$p = 0, \quad z = \eta,$$
$$\eta_t + U\eta_x - \phi_z = 0, \quad z = \eta.$$

In the appendix, it is shown how to simplify this system in the case of long waves of small amplitude. The resulting equation is a KdV equation which differs from the usual KdV equation only in terms of the coefficients. The KdV equation was found for more general shear flows in [15], but we include the derivation in the appendix in order provide an expression for the horizontal velocity which is used in the detection of wave breaking. Denoting the original variables with a hat $\hat{}$, we define non-dimensional variables by

$$\hat{x} = h_0 x,\ \hat{z} = h_0 z,\ \hat{\eta} = h_0\eta,\ \hat{t} = \frac{h_0}{\sqrt{gh_0}}t,$$
$$\hat{u} = \sqrt{gh_0}u \text{ and } \hat{\Gamma} = \frac{\Gamma\sqrt{gh_0}}{h_0}.$$

The KdV equation derived in Appendix B is then written as

$$\eta_t + c_+\eta_x + \frac{c_+(3+\Gamma^2)}{(1+c_+^2)}\eta\eta_x + \frac{c_+^3}{3(1+c_+^2)}\eta_{xxx} = 0. \qquad (1)$$

Moreover, the horizontal velocity can be written as

$$u = c_+\eta + \frac{-1}{2(2c_+ + \Gamma)}\eta^2 + \frac{1+3c_+^2}{6(2c_+ + \Gamma)}\eta_{xx} - c_+\frac{1}{2}(1+z)^2\eta_{xx}.$$

The constant c_+ is defined by $c_+ = \frac{-\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + 1}$. The expression for the horizontal velocity of a fluid particle in the presence of the shear flow is given in non-dimensional variables as

$$U(x, z, t) = c_+\eta + \frac{-1}{2(2c_+ + \Gamma)}\eta^2 + \frac{1+3c_+^2}{6(2c_+ + \Gamma)}\eta_{xx} - c_+\frac{1}{2}(1+z)^2\eta_{xx} + z\Gamma. \qquad (2)$$

Since the horizontal component of the particle velocity has been found, it may be compared to the local phase velocity of the wave. As we mentioned already, this leads to the kinematic breaking criterion. Sections 3 and 4 are dedicated to the application of this convective breaking criterion to the solitary and periodic traveling waves, respectively.

3. Maximum waveheight for solitary waves

We now evaluate the kinematic breaking criterion in the case of the solitary wave solution of the KdV equation (1) with background vorticity. The solitary-wave solution of the KdV equation (1) is given by

$$\eta = H\,\mathrm{sech}^2\left(\sqrt{\frac{(3+\Gamma^2)H}{4(1-c_+\Gamma)}}(x - x_0 - ct)\right).$$

Denoting the argument by $\mathcal{B}(x, t) = \sqrt{\frac{(3+\Gamma^2)H}{4(1-c_+\Gamma)}}(x - x_0 - ct)$, η_{xx} is given by

$$\eta_{xx} = H^2\frac{(3+\Gamma^2)}{4(1-c_+\Gamma)}(-6\,\mathrm{sech}^2(\mathcal{B}) + 4)\,\mathrm{sech}^2(\mathcal{B}),$$

where H is the waveheight, x_0 is the initial location of the wave crest, and $c = c_+ + \frac{(3+\Gamma^2)H}{3(\Gamma+2c_+)}$ is the phase velocity. Since the solitary wave retains its shape for all time, the crest speed is equal to the phase speed of the wave, and we may evaluate at $(x - x_0 - ct) = 0$ in which case we obtain $\eta = H$ and $\eta_{xx} = \frac{-H^2}{2}\frac{(3+\Gamma^2)}{(1-c_+\Gamma)}$. Substituting these values into the horizontal component of the velocity field in the KdV approximation (2) and evaluating at the free surface transforms the breaking criterion into

Table 1
Critical waveheight for solitary-wave solutions of the KdV-equation for various values of the vorticity parameter Γ.

Γ	$H_{\text{max solitary}}$
−0.4	0.8229
−0.3	0.7911
−0.2	0.7578
−0.1	0.7233
0	0.6879
0.1	0.6519
0.2	0.6157
0.3	0.5798
0.4	0.5444

$$c_+H + \frac{-H^2}{2(2c_+ + \Gamma)} + \frac{H^2}{2}\frac{(3+\Gamma^2)}{(1-c_+\Gamma)} \times \left\{ -\frac{1+3c_+^2}{6(2c_+ + \Gamma)} + c_+\frac{1}{2}(1+H)^2 \right\} + H\Gamma \geq c_+ + \frac{(3+\Gamma^2)H}{3(\Gamma + 2c_+)}. \quad (3)$$

To find the critical value of the waveheight, we require equality, and then rearrange terms to obtain

$$\mathbb{P}(H) = \frac{H^4}{4}\frac{(3+\Gamma^2)}{(1-c_+\Gamma)}c_+ + \frac{H^3}{2}\frac{(3+\Gamma^2)}{(1-c_+\Gamma)}c_+ + H^2\left(\frac{-1}{2(2c_+ + \Gamma)} - \frac{3+\Gamma^2}{12(1-\Gamma c_+)}\frac{3c_+^2+1}{2c_+ + \Gamma} + \frac{3+\Gamma^2}{1-\Gamma c_+}c_+ \right) + H\left(c_+ - \frac{3+\Gamma^2}{3(2c_+ + \Gamma)} + \Gamma \right) - c_+ = 0.$$

Contemplating the roots of the fourth-order polynomial in $\mathbb{P}(H)$, we note that the derivative of $\mathbb{P}(H)$ is positive for $H \geq 0$ and $\mathbb{P}(0) < 0$ while $\mathbb{P}(1) > 0$. Therefore, $\mathbb{P}(H)$ can have only one positive root in [0, 1]. For given values of the vorticity Γ, this root can be found numerically to obtain the following values of the maximum admissible waveheight for the solitary wave.

It can be clearly seen from Table 1 that the critical waveheights for the solitary waves are increasing for larger favorable shear flow ($\Gamma < 0$). It is noteworthy that even though the KdV equation admits solutions with any waveheight, solitary waves with a waveheight larger than $H_{\text{max solitary}}$ do not describe actual surface waves since these waves already feature incipient wave breaking.

4. Maximum waveheight for cnoidal waves

The cnoidal wave solutions of the KdV equation (1) with background vorticity are given by

$$\eta = f_2 + (f_1 - f_2)cn^2(\mathcal{B}), \quad (4)$$

where the solution is defined by the three constants f_1, f_2 and f_3 which are arranged in the order $f_3 < f_2 < f_1$, cn is one of the Jacobian elliptic functions defined by the incomplete elliptic integral of the first kind [22], and the modulus of cn is given by $m = (f_1 - f_2)/(f_1 - f_3)$. The argument is $\mathcal{B} = \sqrt{\frac{(3+\Gamma^2)}{4(1-c_+\Gamma)}}(f_1 - f_3)^{1/2}(x - ct)$. The phase speed of the wave is $c = c_+ + \frac{(3+\Gamma^2)}{3(2c_+ + \Gamma)}(f_1 + f_2 + f_3)$, and the wavelength is given by $\lambda = 4\sqrt{\frac{(1-c_+\Gamma)}{(3+\Gamma^2)}}K(m)\frac{1}{\sqrt{f_1 - f_3}}$, where $K(m)$ is the complete elliptic integral of the first kind.

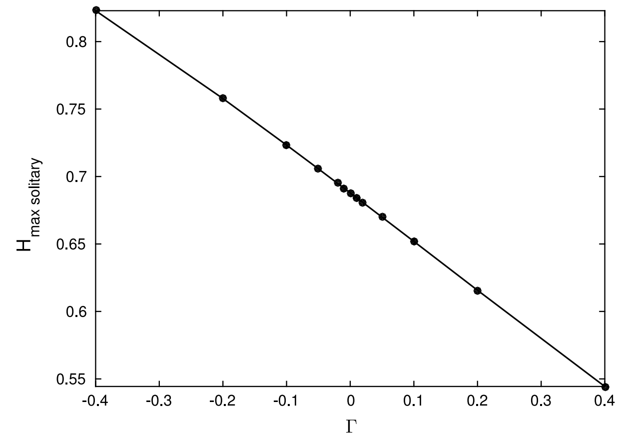

Fig. 2. Plots of maximum admissible waveheight for solitary-wave solutions as a function of Γ for breaking criterion (3).

As shown in [22], the parameters f_1, f_2 and f_3 can be expressed in terms of waveheight $H = f_1 - f_2$ and the elliptic parameter m as follows:

$$f_1 = \frac{H}{m}\left(1 - \frac{E(m)}{K(m)}\right), \quad f_2 = \frac{H}{m}\left(1 - m - \frac{E(m)}{K(m)}\right), \quad f_3 = \frac{H}{m}\left(-\frac{E(m)}{K(m)}\right). \quad (5)$$

Therefore, fixing H and m is sufficient to specify any cnoidal wave as long as the undisturbed water level is set to zero. We now test the kinematic breaking criterion to determine whether a given cnoidal wave is a reasonable description of a water wave in the KdV approximation in the presence of the shear flow. Differentiating (4) twice with respect x yields η_{xx} in the form

$$\eta_{xx} = (f_1 - f_2)(f_1 - f_3)\frac{(3+\Gamma^2)}{2c_+^2} \times \left(sn^2(\mathcal{B})dn^2(\mathcal{B}) - cn^2(\mathcal{B})dn^2(\mathcal{B}) + msn^2(\mathcal{B})cn^2(\mathcal{B}) \right).$$

Evaluating at $(x - ct) = 0$ yields $\eta = f_1$ and $\eta_{xx} = -(f_1 - f_2)(f_1 - f_3)\frac{(3+\Gamma^2)}{2c_+^2}$ and the breaking criterion can be written as

$$c_+f_1 - \frac{f_1{}^2}{2(2c_+ + \Gamma)} + (f_1 - f_2)(f_1 - f_3)\frac{(3+\Gamma^2)}{2c_+^2} \times \left\{ -\frac{1+3c_+^2}{6(2c_+ + \Gamma)} + c_+\frac{1}{2}(1+f_1)^2 \right\} + f_1\Gamma \geq c_+ + \frac{(3+\Gamma^2)(f_1 + f_2 + f_3)}{3(\Gamma + 2c_+)}. \quad (6)$$

In the following, we use the constants

$$a = \frac{1}{m}\left(1 - \frac{E(m)}{K(m)}\right), \quad b = \frac{1}{m}\left(1 - m - \frac{E(m)}{K(m)}\right) \text{ and } c = \frac{1}{m}\left(\frac{E(m)}{K(m)}\right). \quad (7)$$

Table 2
Critical waveheight for the cnoidal solution of the KdV equation, calculated for various values of the elliptic parameter m and with $\Gamma = 0$.

m	$H_{\max\ \text{cnoidal}}$	Wavelength (λ)	α	β	S	Wave speed c
0.01	0.0196	2.591	0.0098	0.1489	0.0661	0.0201
0.1	0.1698	2.857	0.0849	0.1224	0.6933	0.1951
0.2	0.2909	3.178	0.1454	0.0990	1.4689	0.3516
0.3	0.3820	3.507	0.1910	0.0812	2.3499	0.4715
0.4	0.4548	3.849	0.2274	0.0674	3.3702	0.5669
0.5	0.5152	4.218	0.2575	0.0562	4.5834	0.6469
0.6	0.5667	4.632	0.2833	0.0465	6.0813	0.7176
0.7	0.6114	5.128	0.3056	0.0380	8.0399	0.7839
0.8	0.6504	5.781	0.3252	0.0299	10.869	0.8511
0.9	0.6841	6.829	0.3420	0.0214	15.952	0.9295

Table 3
Critical waveheight for the cnoidal solution of the KdV equation, calculated for various values of the elliptic parameter m and with $\Gamma = -0.1$.

m	$H_{\max\ \text{cnoidal}}$	λ	α	β	S	c
0.01	0.0207	2.6539	0.0103	0.1420	0.0728	0.0202
0.1	0.1791	2.9202	0.0896	0.1173	0.7637	0.2005
0.2	0.3070	3.2467	0.1535	0.0949	1.6180	0.3655
0.3	0.4031	3.5835	0.2016	0.0779	2.5883	0.4925
0.4	0.4796	3.9344	0.2398	0.0646	3.7121	0.5936
0.5	0.5431	4.3119	0.2715	0.0538	5.0485	0.6783
0.6	0.5971	4.4766	0.2986	0.0446	6.6983	0.7531
0.7	0.6440	5.2443	0.3220	0.0364	8.8557	0.8231
0.8	0.6849	5.9126	0.3425	0.0286	11.9720	0.8941
0.9	0.7201	6.9854	0.3601	0.0205	17.5701	0.9769

Table 4
Critical waveheight for the cnoidal solution of the KdV equation, calculated for various values of the elliptic parameter m and with $\Gamma = 0.1$.

m	$H_{\max\ \text{cnoidal}}$	λ	α	β	S	c
0.01	0.0187	2.5269	0.0093	0.1566	0.0596	0.0200
0.1	0.1604	2.7923	0.0802	0.1283	0.6253	0.1894
0.2	0.2746	3.1063	0.1373	0.1036	1.3248	0.3379
0.3	0.3609	3.4273	0.1804	0.0851	2.1193	0.4510
0.4	0.4298	3.7609	0.2149	0.0707	3.0395	0.5411
0.5	0.4870	4.1200	0.2435	0.0589	4.1337	0.6167
0.6	0.5359	4.5240	0.2680	0.0489	5.4845	0.6836
0.7	0.5784	5.0073	0.2892	0.0399	7.2510	0.7464
0.8	0.6155	5.6437	0.3078	0.0314	9.8027	0.8100
0.9	0.6475	6.6660	0.3238	0.0225	14.3864	0.8844

Substituting these definitions into Eq. (6) and setting the left and right hand sides equal yields

$$\begin{aligned}\mathbb{Q}(H) &= H^4(a^4 + a^3c - a^3b - a^2bc)\frac{(3+\Gamma^2)}{(4c_+)} \\ &+ H^3(a^3 + a^2c - a^2b - abc)\frac{(3+\Gamma^2)}{2c_+} \\ &+ H^2\left(\frac{-a^2}{2(2c_+ + \Gamma)} + \left\{-\frac{3+\Gamma^2}{12c_+^2}\frac{3c_+^2+1}{2c_+ + \Gamma} + \frac{3+\Gamma^2}{4c_+}\right\}\right. \\ &\left.\times\,(a^2 + ac - ab - bc)\right) \\ &+ H\left(c_+a - \frac{3+\Gamma^2}{3(2c_+ + \Gamma)(a+b-c)} + \Gamma a\right) - c_+ = 0.\end{aligned} \tag{8}$$

Here $\mathbb{Q}(H)$ is a fourth-order polynomial in H, and by fixing the values of m and Γ (8) can be solved numerically in the interval [0, 1] to obtain the maximum admissible waveheight for the cnoidal wave, $H_{\max\ \text{cnoidal}}(m, \Gamma)$.

Tables 2, 3 and 4 list the values of m and the corresponding $H_{\max\ \text{cnoidal}}$ for three different vorticity values $\Gamma = 0, -0.1$ and 0.1. This is in agreement with the results of [23] in the absence of shear flow. These tables also list the corresponding values of λ, α, β and Stokes number S defined to be the ratio α/β. Fig. 3 shows a plot of maximum allowable waveheight of the cnoidal wave as a function of elliptic parameter m for different vorticity constants $\Gamma = 0.1, 0$ and -0.1. It can be seen that $H_{\max\ \text{cnoidal}}$ approaches zero as m approaches zero, i.e. in the linear limit. The maximum allowable waveheight of the cnoidal wave as a function of wavelength (λ) for the values $\Gamma = 0.1, 0$ and -0.1 are shown in Fig. 4.

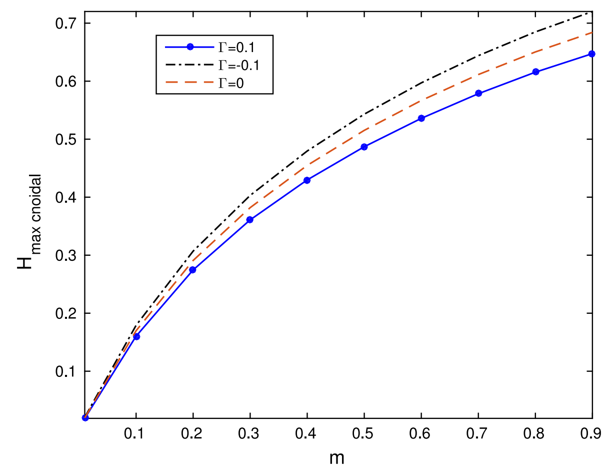

Fig. 3. Maximum allowable waveheight for the cnoidal solution as a function of the elliptic parameter m for $\Gamma = 0.1, 0, -0.1$.

Fig. 5 shows a plot of maximum allowable waveheight of the cnoidal wave as a function of Γ for different elliptic parameters $m = 0.1$ and $m = 0.8$. It can be seen from Fig. 5 that for favorable case of shear flow ($\Gamma \le 0$) the maximum allowable waveheight for the cnoidal solution is increasing with larger values of vorticity. Similar patterns can be found for other elliptic parameters m between 0 and 1.

Fig. 6 shows the wavelength corresponding to the maximum allowable waveheight for the critical cnoidal solution as a function of Γ for elliptic parameters $m = 0.1$ and $m = 0.8$. This figure shows that negative Γ leads to longer critical wavelength and positive Γ leads to shorter critical wavelength.

5. Conclusion

In this article, a derivation of the KdV equation as a model for surface water waves in the presence of a background shear flow has been given. The derivation also led to expressions for the velocity field in terms of the dependent variable $\eta(x, t)$. Using the horizontal component of the velocity field, solitary and cnoidal wave solutions were examined with respect to possible incipient wave breaking. Concerning the solitary wave solutions, the computations have been performed for various values of the

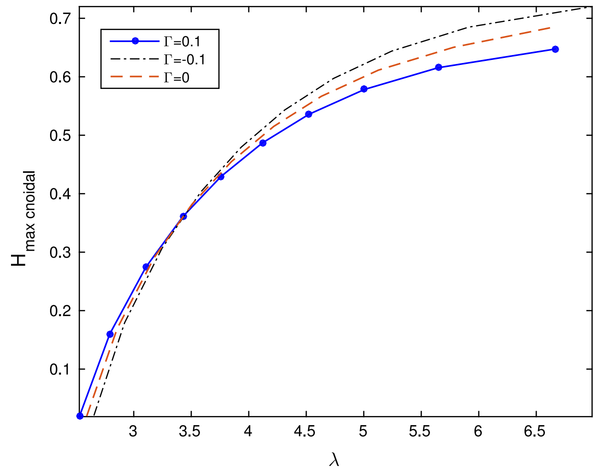

Fig. 4. Maximum allowable waveheight for the cnoidal solution as a function of wavelength λ for $\Gamma = 0.1, 0, -0.1$.

vorticity constant Γ, and the critical waveheight for solitary waves are tabulated in Table 1. It can be seen from Fig. 2 that the critical waveheights are increasing with large favorable shear ($\Gamma < 0$), and decreasing with large unfavorable shear ($\Gamma < 0$).

For cnoidal waves, the onset of breaking can occur at rather small waveheights. Indeed, if the elliptic parameter m is small enough, wave breaking may already occur at nondimensional waveheights less than 0.2, and with Stokes number near 1 (see Tables 2, 3 and 4. These results are similar for zero or small values of background shear Γ, but the effect of the background shear is similar to the case of the solitary wave in that it can either facilitate or inhibit wave breaking depending on the sign of Γ. If the vorticity is defined as $\omega = W_x - U_z = -\Gamma$, it can be seen that this finding is in qualitative agreement with the numerical study [17] where stagnation points of the full Euler equations were sought.

The results of Section 4 also suggest that in the limit as the elliptic parameter m approaches 1 the wave breaking condition in the cnoidal waves approaches the condition for the solitary wave. On the other hand, for the linear case ($m \to 0$), the cnoidal waves have negligible waveheight, and the critical waveheight in this case indeed tends to 0 for all values of the vorticity constants Γ.

The main finding of the paper is that the presence of a background shear current may lead to incipient wave breaking for smaller waveheights than in the irrotational case. It may be possible to incorporate vorticity into the description of surface waves

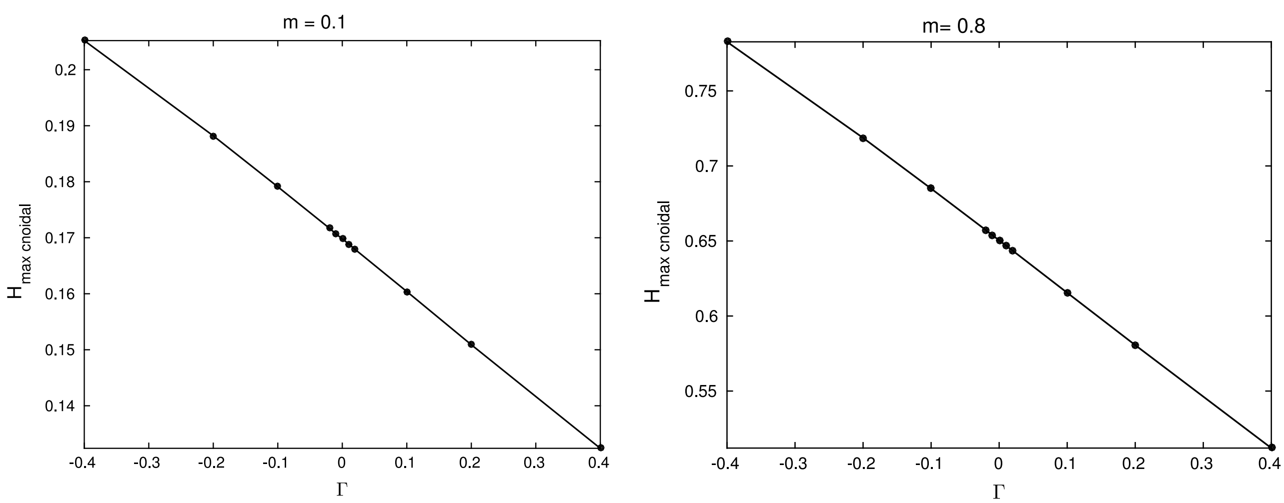

Fig. 5. The left panel shows the maximum allowable waveheight for the cnoidal solution as a function of Γ for $m = 0.1$. The right panel shows the maximum allowable waveheight for the cnoidal solution as a function of Γ for $m = 0.8$.

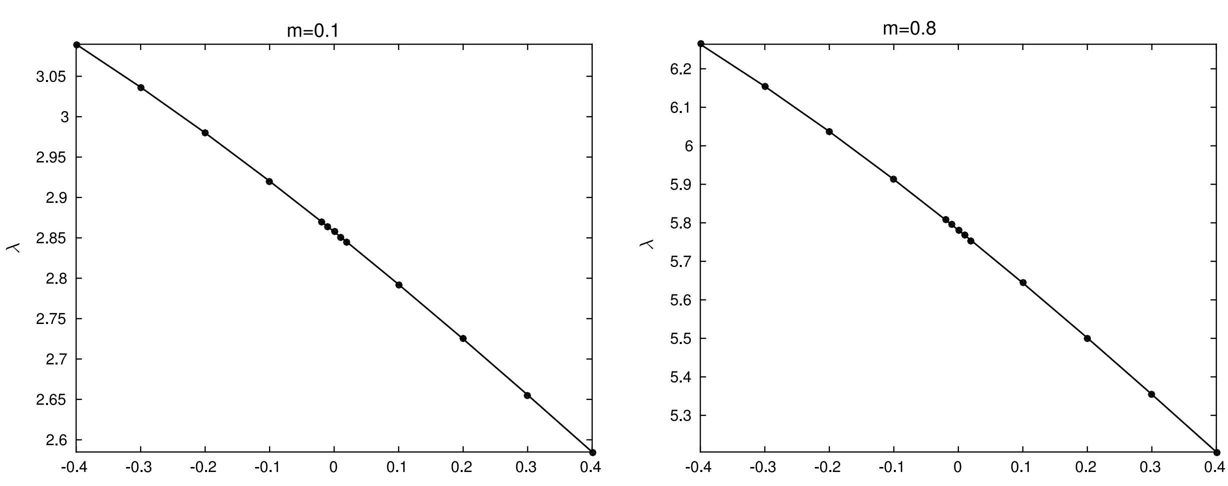

Fig. 6. The left panel shows the wavelength corresponding to the maximum allowable waveheight for the cnoidal solution as a function of Γ for $m = 0.1$. The right panel shows the wavelength corresponding to the maximum allowable waveheight for the cnoidal solution as a function of Γ for $m = 0.8$.

by Boussinesq or full Euler models in such a way to as match the tightened breaking criteria advocated in [19,20]. This will be the subject of further work.

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Appendix A. Derivation of a Boussinesq system with constant vorticity

In the free-surface wave problem laid out in Section 2, the total velocity field satisfies the equation

$$\mathbf{u}_t + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times(\nabla\times\mathbf{u}) + g\mathbf{e}_z = 0, \qquad -h_0 < z = \eta.$$

Defining the vorticity as usual by $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, and using the fact that the background flow is time-independent, the equation can be written as

$$\nabla\phi_t + \frac{1}{2}\nabla|\mathbf{u}|^2 + g\mathbf{e}_z = \mathbf{u}\times\boldsymbol{\omega}, \qquad -h_0 < z = \eta.$$

As the left-hand side is obviously a gradient, the term $\mathbf{u}\times\boldsymbol{\omega}$ must also be the gradient of a function. Following [24] it becomes plain that $\mathbf{u}\times\boldsymbol{\omega} = \nabla G$, where the function G is given by

$$G = -\Gamma\int_{-h_0}^{z}\frac{\partial\phi}{\partial x}\,dz - \frac{\Gamma^2}{2}z^2.$$

In order to bring out the difference in scales, non-dimensional variables are introduced as follows:

$$\tilde{x} = \frac{x}{l}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{t} = \frac{\sqrt{gh_0}t}{l},$$
$$\text{and} \quad \tilde{\Gamma} = \frac{\Gamma h_0}{\sqrt{gh_0}} \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{\phi} = \frac{h_0}{al\sqrt{gh_0}}\phi,$$

where tilde ˜ denotes non-dimensional variables and h_0, l and a denote a characteristic water depth, wavelength and wave amplitude, respectively. The parameter $\alpha = a/h_0$ represents an amplitude to depth ratio, and the parameter $\beta = h_0^2/l^2$ represents a water depth to wavelength ratio. As a result of the scaling the governing equations and boundary conditions for the irrotational wave problem read

$$\beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0 \quad -1 < \tilde{z} < \alpha\tilde{\eta} \tag{9a}$$

$$\tilde{\phi}_{\tilde{z}} = 0 \quad \text{at } \tilde{z} = -1 \tag{9b}$$

$$\beta\left\{\frac{\partial\tilde{\eta}}{\partial\tilde{t}} + \left[\alpha\tilde{\eta}\tilde{\Gamma} + \alpha\frac{\partial\tilde{\phi}}{\partial\tilde{x}}\right]\frac{\partial\tilde{\eta}}{\partial\tilde{x}}\right\} = \tilde{\phi}_{\tilde{z}} \quad \text{at } \tilde{z} = \alpha\tilde{\eta} \tag{9c}$$

$$\beta(\tilde{\phi}_{\tilde{t}} + \tilde{\eta}) + \frac{\alpha\beta}{2}\left[\tilde{\phi}_{\tilde{x}} + \frac{\tilde{\Gamma}}{\alpha}(\alpha\tilde{\eta})\right]^2 + \frac{\alpha}{2}\left[\frac{\partial\tilde{\phi}}{\partial z}\right]^2 + \frac{\beta}{ag}G = 0 \quad \text{at } \tilde{z} = \alpha\tilde{\eta} \tag{9d}$$

Now we expand the velocity potential $\tilde{\phi}$ as an asymptotic series in the vertical coordinate. We choose the expansion

$$\tilde{\phi} = \sum_{n=0}^{\infty}(1+\tilde{z})^n\phi_n. \tag{10}$$

From Eqs. (9a), (9b) and (10), we have

$$\tilde{\phi} = \phi_0 - \frac{\beta}{2}(1+\tilde{z})^2\frac{\partial^2\phi_0}{\partial\tilde{x}^2} + \frac{\beta^2}{24}(1+\tilde{z})^4\frac{\partial^4\phi_0}{\partial\tilde{x}^4} + \mathcal{O}(\beta^3), \tag{11}$$

which is a series solution with only one unknown function ϕ_0. We use the following procedure to find a Boussinesq system of two evolution equations. We first substitute the asymptotic expression for $\tilde{\phi}$ in the boundary conditions Eqs. (9c) and (9d), and then collect all terms of zeroth and first order in α and β. We then differentiate the dynamic boundary condition with respect to $\tilde{x}$ and express the boundary conditions in terms of the non-dimensional horizontal velocity at the bottom $\frac{\partial\phi_0}{\partial\tilde{x}} = \tilde{v}$. Using these procedures results in the system

$$\begin{aligned}
&\tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} + \alpha\tilde{\Gamma}\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{v})_{\tilde{x}} - \frac{\beta}{6}\frac{\partial^3\tilde{v}}{\partial\tilde{x}^3} = \mathcal{O}(\alpha\beta, \beta^2),\\
&\tilde{v}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{v}\tilde{v}_{\tilde{x}} - \tilde{\Gamma}\tilde{v}_{\tilde{x}} - \frac{\beta}{2}\frac{\partial^3\tilde{v}}{\partial\tilde{x}^2\partial\tilde{t}} + \frac{\beta}{6}\tilde{\Gamma}\frac{\partial^3\tilde{v}}{\partial\tilde{x}^3} = \mathcal{O}(\alpha\beta, \beta^2).
\end{aligned} \tag{12}$$

Disregarding term of $\mathcal{O}(\alpha\beta, \beta^2)$ yields the Boussinesq system.

Appendix B. Derivation of the KdV equation with constant vorticity

The Korteweg–de Vries equation is derived from the system (12) by specializing to a wave moving to the right. To lowest order, neglecting the terms of order α and β, the system (12) is

$$\tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} = 0, \tag{13}$$

$$\tilde{v}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} - \tilde{\Gamma}\tilde{v}_{\tilde{x}} = 0. \tag{14}$$

The system can be diagonalized by introducing characteristic coordinates. Indeed, we introduce the new variables r and s defined by $(r, s) = P^{-1}(\tilde{\eta}, \tilde{v})$ where

$$P^{-1} = \frac{1}{1+\tilde{c}_+^2}\begin{pmatrix}1 & \tilde{c}_+\\ \tilde{c}_+ & -1\end{pmatrix}. \tag{15}$$

With $\tilde{c}_+ = \frac{-\tilde{\Gamma}}{2} + \sqrt{\frac{\tilde{\Gamma}^2}{4}+1}$, and $\tilde{c}_- = \frac{-\tilde{\Gamma}}{2} - \sqrt{\frac{\tilde{\Gamma}^2}{4}+1}$ as the conjugate of $\tilde{c}_+$, the solutions of system (13) are

$$\begin{aligned}
\tilde{v}(x,t) &= \frac{1}{1+\tilde{c}_+^2}\Big[\tilde{c}_+\tilde{\eta}_0(x-\tilde{c}_+t) + \tilde{c}_+^2\tilde{v}_0(x-\tilde{c}_+t)\\
&\quad - \tilde{c}_+\tilde{\eta}_0(x-\tilde{c}_-t) + \tilde{v}_0(x-\tilde{c}_-t)\Big],\\
\tilde{\eta}(x,t) &= \frac{1}{1+\tilde{c}_+^2}\Big[\tilde{\eta}_0(x-\tilde{c}_+t) + \tilde{c}_+\tilde{v}_0(x-\tilde{c}_+t)\\
&\quad + \tilde{c}_+^2\tilde{\eta}_0(x-\tilde{c}_-t) - \tilde{c}_+\tilde{v}_0(x-\tilde{c}_-t)\Big].
\end{aligned} \tag{16}$$

The unidirectional KdV equation is derived from the system (12) by specializing to a wave moving to the right with speed $\tilde{c}_+$. We look for a solution, correct to first order in α and β, in the form

$$\tilde{v} = \tilde{c}_+\tilde{\eta} + \alpha A + \beta B + \mathcal{O}(\alpha\beta, \beta^2),$$

where A and B are functions of $\tilde{\eta}$ and its $\tilde{x}$ derivatives. Then the system (12) becomes

$$\begin{aligned}
&\tilde{c}_+\tilde{\eta}_{\tilde{t}} + \tilde{c}_+^2\tilde{\eta}_{\tilde{x}} + \alpha(\tilde{c}_+A_x + 2\tilde{c}_+^2\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \tilde{\Gamma}\tilde{c}_+\tilde{\eta}\tilde{\eta}_{\tilde{x}})\\
&\quad + \beta(\tilde{c}_+B_{\tilde{x}} - \frac{1}{6}\tilde{c}_+^2\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}}) = \mathcal{O}(\alpha\beta, \beta^2),\\
&\tilde{c}_+\tilde{\eta}_{\tilde{t}} + \tilde{c}_+^2\tilde{\eta}_{\tilde{x}} + \alpha(A_{\tilde{t}} + \tilde{c}_+^2\tilde{\eta}\tilde{\eta}_{\tilde{x}} - \tilde{\Gamma}A_{\tilde{x}})\\
&\quad + \beta(B_{\tilde{t}} - \frac{1}{2}\tilde{c}_+\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} + \frac{\tilde{\Gamma}}{6}\tilde{c}_+\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}}) = \mathcal{O}(\alpha\beta, \beta^2).
\end{aligned}$$

Since $\tilde{\eta}_{\tilde{t}} = -\tilde{c}_+\tilde{\eta}_{\tilde{x}}$, all derivatives in the first order terms may be replaced by $-\tilde{c}_+$ times the x derivatives. Then the two equations are consistent if

$$A = \frac{-1}{2(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}^2, \quad B = \frac{1+3\tilde{c}_+^2}{6(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}_{\tilde{x}\tilde{x}}.$$

Hence we have the expression

$$\tilde{v} = \tilde{c}_+\tilde{\eta} + \alpha\frac{-1}{2(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}^2 + \beta\frac{1+3\tilde{c}_+^2}{6(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}_{\tilde{x}\tilde{x}} + \mathcal{O}(\alpha\beta, \beta^2), \quad (17)$$

and the equation

$$\tilde{\eta}_{\tilde{t}} + \tilde{c}_+\tilde{\eta}_{\tilde{x}} + \alpha\frac{\tilde{c}_+(3+\tilde{\Gamma}^2)}{(1+\tilde{c}_+^2)}\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \beta\frac{\tilde{c}_+^3}{3(1+\tilde{c}_+^2)}\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2).$$

From Eq. (11), the non-dimensional horizontal velocity becomes

$$\tilde{u} = \tilde{v} - \frac{\beta}{2}(1+\tilde{z})^2\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \mathcal{O}(\beta^2). \quad (18)$$

From Eqs. (17) and (18), we have

$$\tilde{u} = \tilde{c}_+\tilde{\eta} + \alpha\frac{-1}{2(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}^2 + \beta\frac{1+3\tilde{c}_+^2}{6(2\tilde{c}_+ + \tilde{\Gamma})}\tilde{\eta}_{\tilde{x}\tilde{x}} - \tilde{c}_+\frac{\beta}{2}(1+\tilde{z})^2\tilde{\eta}_{\tilde{x}\tilde{x}} + \mathcal{O}(\alpha\beta, \beta^2). \quad (19)$$

After neglecting the second-order terms, the KdV equation and the second-order expression for the horizontal component of the velocity field appear.

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